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# A Chern-Simons Effective Field Theory for the Pfaffian Quantum Hall State


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We present a low-energy effective field theory describing the universality class of the Pfaffian quantum Hall state. To arrive at this theory, we observe that the edge theory of the Pfaffian state of bosons at $\nu = 1$ is an $SU(2)_2$ Kac-Moody algebra. It follows that the corresponding bulk effective field theory is an $SU(2)$ Chern-Simons theory with coupling constant $k = 2$. The effective field theories for other Pfaffian states, such as the fermionic one at $\nu = 1/2$ are obtained by a flux-attachment procedure. We discuss the non-Abelian statistics of quasiparticles in the context of this effective field theory.


## I. INTRODUCTION

Recently there has been considerable interest in a new class of quantum Hall states, combinining aspects of BCS pairing with Laughlin-type ordering [1–6]. The states appear to be incompressible, and to exhibit non-Abelian statistics. Their properties have mainly been inferred by extrapolation in quantum statistics (of electrons) from ordinary superconductors [2], from numerical studies [2,6], and from the existence of attractive trial wave functions [1–5]. To assure ourselves of the robustness of these properties, and for calculational purposes, it is important to have a low-energy, long-wavelength effective field theory. In this paper, we shall supply such a theory.

Analyses of the ground states of the lowest Landau level Hamiltonian with three-body interactions of the form

$$H = \sum_{i>j>k} \delta'(z_i - z_j)\delta'(z_i - z_k) \tag{1}$$

have unearthed a number of fascinating properties of these states [1–5]. In particular, at Landau level filling fraction $\nu = 1/2$, the ground state is the so-called Pfaffian state,

$$\Psi_{\rm Pf} \;=\; {\rm Pf}\left(\frac{1}{z_i - z_j}\right) \prod_{i>j}(z_i - z_j)^2 \;\, e^{-\frac{1}{4\ell_0^2}\sum |z_i|^2} \tag{2}$$

(where ${\rm Pf}\left(\frac{1}{z_i-z_j}\right) = \mathcal{A}\left(\frac{1}{z_1-z_2}\,\frac{1}{z_3-z_4}\,\ldots\right)$ is the antisymmetrized product over pairs of electrons). There is a gap to all excited states and a discontinuity in the chemical potential, so (2) describes an incompressible state with quantized Hall conductance. This state (or a state in the same universality class, in a sense to be discussed below) might describe the plateaus observed at $\nu = 5/2$ in a single-layer system and $\nu = 1/2$ in double-layer systems.

It will be useful to consider a slight generalization of (2):

$$\Psi_{\rm Pf} \;=\; {\rm Pf}\left(\frac{1}{z_i - z_j}\right) \prod_{i>j}(z_i - z_j)^q \;\, e^{-\frac{1}{4\ell_0^2}\sum |z_i|^2} \tag{3}$$

For $q$ even, (3) is a state of fermions at $\nu = 1/q$; for $q$ odd, (3) is a state of bosons. The bosonic state with $q = 1$ is, in fact, an approximate ground state at $\nu = 1$ of bosons interacting through the Coulomb interaction [6].

As charge is removed from the Pfaffian state (2), quasiholes are created. The quasiholes carry half of a flux quantum, as may be seen from the following state, which has quasiholes at $\eta_1$ and $\eta_2$:

$$\Psi_{\rm Pf} \;=\; {\rm Pf}\left(\frac{(z_i - \eta_1)\,(z_j - \eta_2) + i \leftrightarrow j}{z_i - z_j}\right) \prod_{i>j}(z_i - z_j)^2 \;\, e^{-\frac{1}{4\ell_0^2}\sum |z_i|^2} \tag{4}$$

The most remarkable features of the Pfaffian state are exhibited when there are at least four quasiholes. A state with 4 quasiholes at points $\eta_\alpha$ is obtained by modifying the Pfaffian as follows:

$${\rm Pf}\left(\frac{1}{z_j - z_k}\right) \rightarrow {\rm Pf}\left(\frac{(z_j - \eta_1)(z_j - \eta_2)(z_k - \eta_3)(z_k - \eta_4) + (j \leftrightarrow k)}{z_j - z_k}\right) \equiv {\rm Pf}_{(12)(34)}\ . \tag{5}$$

However, this is not the only four-quasihole state with quasiholes at $\eta_1, \eta_2, \eta_3, \eta_4$. $\mathrm{Pf}_{(13)(24)}$ and $\mathrm{Pf}_{(14)(23)}$ seem to be equally good. In fact, only two of these three are linearly independent, as a consequence of the identity [3]:

$$\mathrm{Pf}_{(12)(34)} - \mathrm{Pf}_{(14)(23)} = \frac{\eta_{14}\eta_{23}}{\eta_{13}\eta_{24}} \left( \mathrm{Pf}_{(12)(34)} - \mathrm{Pf}_{(13)(24)} \right) . \tag{6}$$

It can similarly be shown that there are $2^{n-1}$ linearly independent states of $2n$ quasiholes at fixed positions $\eta_1, \eta_2, \ldots, \eta_{2n}$ [3]. This exponential degeneracy has been interpreted in terms of the possible occupation numbers of $n$ fermionic zero modes associated with the quasiholes [4–6]. As follows from the numerical value of the degeneracy, the above mentioned zero modes are Majorana fermions. We will see below that these fermions appear as edge states in a system of finite size.

The degeneracy of the multi-quasihole states allows for the possibility of non-Abelian statistics. This possibility is, indeed, realized, and the braiding matrices are associated with the group $SO(2n)$ [3]. This remarkable relationship is possible because $SO(2n)$ has a $2^{n-1}$ dimensional spinor representation. Here we again encounter Majorana fermions since the spinor representation is constructed from 2n real $\gamma$-matrices satisfying the fermionic anticommutation relations

$$\{\gamma_i, \gamma_j\} = \delta_{i,j} \tag{7}$$

The operation of braiding the $i^{th}$ and $j^{th}$ quasiholes corresponds (up to a phase) to the $SO(2n)$ rotation by $\pi$ in the $i-j$ plane. In other words, the action of this braiding operation on the space of $2n$ quasihole states is given by the matrix which represents the corresponding $SO(2n)$ rotation in its spinor representation. For instance, an exchange of $\eta_1$ and $\eta_3$ leads to the following rotation between two four-quasihole states (in a particular basis of the four-quasihole states whose definition is unimportant here)

$$\frac{e^{i\pi\left(\frac{1}{8}+\frac{1}{4q}\right)}}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \tag{8}$$

This is precisely the matrix in the spinor representation of $SO(4)$ which represents a rotation by $\pi$ in the $1-3$ plane.

Another important topological quantum number characterizing a quantum Hall state is the ground state degeneracy on a torus. At $\nu = 1/q$, there is always a trivial $q$-fold center-of-mass degeneracy coming from the $\theta$-function generalization of the Jastrow factor. The Pfaffian state has ground state degeneracy $3q$, with the additional factor of 3 coming from the toroidal version of the Pfaffian:

$$\mathrm{Pf}\left(\frac{1}{z_i - z_j}\right) \to \mathrm{Pf}\left(\frac{\theta_a(z_i - z_j)}{\theta_1(z_i - z_j)}\right) \tag{9}$$

where $a = 2, 3, 4$.

The edge excitations of the Pfaffian state take the form [4,5]:

$$\mathcal{A}\left( {z_1}^{p_1} \ldots {z_k}^{p_k} \frac{1}{z_{k+1} - z_{k+2}} \frac{1}{z_{k+3} - z_{k+4}} \ldots \right) \; S(z_1, \ldots, z_N) \; \prod_{i>j} (z_i - z_j)^2 \;\; e^{-\frac{1}{4\ell_0^2}\sum |z_i|^2} \tag{10}$$

The arbitrary distinct positive integers $p_1, \ldots, p_k$ correspond to the creation of neutral fermionic excitations; this is the $c = 1/2$ sector of the edge theory. $S(z_1, \ldots, z_N)$ is a symmetric polynomial; this modification of the ground state corresponds to the creation of chiral bosonic excitations in the $c = 1$ charged sector of the edge theory. At $\nu = 1/2$, the compactification radius of the boson is $R = 1/\sqrt{2}$. It can be shown that these excitations span the Hilbert space of the edge theory; hence the edge theory has total central charge $c = \frac{1}{2} + 1$.

The specific form of the trial wavefunctions considered above played a crucial role in the determination of the properties of the Pfaffian state, such as the non-Abelian braiding statistics. Of course, we would like to believe that the Pfaffian state is a representative of an entire universality class of states which have the same ‘topological properties’, such as braiding statistics and ground state degeneracy on the torus. To investigate the existence and stability of this universality class, we need a low-energy, long-wavelength effective field theory for the Pfaffian state.

An effective field theory for the Laughlin states at $\nu = 1/(2k+1)$ was obtained by a Landau-Ginzburg construction

$$\begin{aligned} \mathcal{L}_{\text{eff}} = \psi^* \left(i\partial_0 - (a_0 + A_0)\right)\psi + \frac{\hbar^2}{2m^*}\, \psi^* (i\nabla - (\mathbf{a} + \mathbf{A}))^2 \psi + u|\psi|^4 \\ + \frac{1}{2k+1}\,\frac{1}{4\pi}\epsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho \end{aligned} \tag{11}$$

The fundamental objects in the Landau-Ginzburg theory are auxiliary bosons, $\psi$, which are electrons with fictitious flux attached via $\mathbf{a}$. The fractionally charged, anyonic quasiparticles are vortices. In a dual theory, which results from integrating out the auxiliary bosons, the quasiparticles are fundamental:

$$\mathcal{L}_{\rm dual} = \frac{2k+1}{4\pi}\epsilon^{\mu\nu\rho}\alpha_\mu\partial_\nu\alpha_\rho + A_\mu\epsilon^{\mu\nu\rho}\partial_\nu\alpha_\rho + \alpha_\mu j^\mu_{\rm vortex} \tag{12}$$

This dual theory has a simple relationship with the edge theory, and seems more amenable to a non-Abelian generalization.

We do not know what the correct Landau-Ginzburg theory is for the Pfaffian state (see, however, our comments at the end). However, we do know that the edge theory is the conformal field theory of a Majorana fermion and a free boson, with $c = \frac{1}{2} + 1$. We will use this edge theory to deduce the effective field theory of the bulk (which is dual to the Landau-Ginzburg theory). Our first step will be to show how this is done in the simplest case (for reasons which will become clear) of a Pfaffian state of bosons at $\nu = 1$. We will then check that this effective field theory reproduces the bulk properties exhibited by the wavefunctions. Finally, we will comment on the stability of the state.

## II. EFFECTIVE FIELD THEORY OF THE BOSONIC PFAFFIAN STATE AT $\nu = 1$

The edge theory of the bosonic Pfaffian state at $\nu = 1$ has $c = \frac{1}{2} + 1$, but the compactification radius of the $c = 1$ sector is $R = 1$. A marvelous feature of the free bosonic theory at $R = 1$ is that it can be fermionized. The boson, $\phi$, can be replaced by a Dirac fermion, $\psi$, or, equivalently, two Majorana fermions, $\chi_1, \chi_2$:

$$e^{i\phi} = \psi = \chi_1 + i\chi_2 \tag{13}$$

Hence, the edge theory is a theory of a triplet of Majorana fermions, with central charge $c = 3/2$.

The triplet of Majorana fermions tranform under the spin-1 representation of $SU(2)$. Let us call our third Majorana fermion $\chi_3$. The currents (the $T^a$'s are $SU(2)$ generators in the spin-1 representation),

$$J^a = \chi_i (T^a)_{ij} \chi_j \tag{14}$$

form an $SU(2)$ Kac-Moody algebra at level $k = 2$.

$$J^a(z)J^b(0) \sim \frac{2\delta^{ab}}{z^2} + \frac{f^{abc}J^c(0)}{z} + \ldots \tag{15}$$

The $U(1)$ subgroup of $SU(2)$ which is generated by $J^3$ is the $U(1)$ of electric charge since $J^3 = i\chi_1\chi_2 = i\partial\phi$.

This Kac-Moody algebra has a bosonic incarnation as the *chiral sector* of the $SU(2)$ WZW model at $k = 2$:

$$S = \frac{2}{16\pi}\int d^2x\, tr\left(g^{-1}\partial_\mu g\, g^{-1}\partial^\mu g\right) + \frac{2}{24\pi}\int d^3x\, \epsilon^{\mu\nu\lambda}\, tr\left(g^{-1}\partial_\mu g\, g^{-1}\partial_\nu g\, g^{-1}\partial_\lambda g\right) \tag{16}$$

where $g$ is an $SU(2)$-valued matrix field. The $SU(2)$ WZW model has $c = \frac{3k}{k+2} = 3/2$; which is the central charge of a triplet of Majorana fermions which indicates the equivalency between these two models. Hence, we may take the chiral sector of the $SU(2)_2$ WZW model as the edge theory of the bosonic Pfaffian state at $\nu = 1$.

Now, we are most of the way home because the bulk theory corresponding to the chiral $SU(2)_2$ WZW model is simply the $SU(2)$ Chern-Simons theory with Chern-Simons coefficient $k = 2$ [7],

$$S = \frac{2}{4\pi}\int d^3x\, \epsilon^{\mu\nu\lambda}\left(a^a_\mu\partial_\nu a^a_\lambda + \frac{2}{3}f_{abc}a^a_\mu a^b_\nu a^c_\lambda\right) \tag{17}$$

where $a^a_\mu$ is an $SU(2)$ gauge field. An alternative approach begins with the coset construction of the $c = 1/2$ theory [9,10], but it is less transparent, so we do not pursue it further.

## III. TOPOLOGICAL PROPERTIES IN THE EFFECTIVE FIELD THEORY

We can now check that the effective field theory (17) predicts the same topological properties as the analysis of trial wavefunctions. Let us first consider the ground state degeneracy on a torus, which we know to be three for bosons at $\nu = 1$. The Hilbert space of the Chern-Simons theory (17) (since the Chern-Simons Hamiltonian vanishes, all states are ground states) on a torus can be obtained in the following way [7,8]. Consider the path integral of (17) over the three-dimensional region $M$ enclosed by a torus, $\partial M = T^2$,

$$\Psi[a] = \int_{\alpha_{|T^2}=a} D\alpha \; e^{2\int_M d^3x\, \epsilon^{\mu\nu\lambda}\left(\alpha^a_\mu \partial_\nu \alpha^a_\lambda + \frac{2}{3} f_{abc}\alpha^a_\mu \alpha^b_\nu \alpha^c_\lambda\right)} \tag{18}$$

subject to the condition that the gauge field $\alpha^a_\mu$ is equal to a prescribed gauge field $a^a_\mu$ at the toroidal boundary of $M$. This is a state $\Psi[a]$ in the Hilbert space of (17). Other states can be obtained by inserting a Wilson loop in the path integral. Only a Wilson loop that is non-contractable in $M$ – there is only one such topologically distinct loop – will give a non-trivial contribution. Furthermore, at level $k = 2$, only Wilson loops in the $SU(2)$ representations $j = 0, 1/2, 1$ will give non-vanishing path integrals [7]. Hence, we have the three ground states on the torus:

$$\Psi_j[a] = \int_{\alpha_{|T^2}=a} D\alpha \;\; \mathrm{Tr}_j \left\{ \mathcal{P} e^{i\oint a} \right\} \;\; e^{\frac{2}{4\pi}\int_M d^3x\, \epsilon^{\mu\nu\lambda}\left(\alpha^a_\mu \partial_\nu \alpha^a_\lambda + \frac{2}{3} f_{abc}\alpha^a_\mu \alpha^b_\nu \alpha^c_\lambda\right)} \tag{19}$$

where $\mathrm{Tr}_j$ is the trace in the spin $j$ representation of $SU(2)$ and $\mathcal{P}$ denotes path-ordering.

To obtain the degeneracy of the $2n$ quasihole states, we need to first observe that the half-flux-quantum quasiholes carry the spin-1/2 representation of $SU(2)$. Let's consider the four quasihole case; the extension to $2n$ quasiholes is straightforward. We would like the Hilbert space of (17) with four external charges carrying the spin-1/2 representation of $SU(2)$. At large $k$ (weak coupling), it is clear that these charges do not interact except through the constraint that the state be a total $SU(2)$ singlet. There are two different spin-singlet states which can be made with four spin-1/2's: $(|+\rangle|-\rangle - |-\rangle|+\rangle)(|+\rangle|-\rangle - |-\rangle|+\rangle)$ and $|+\rangle|+\rangle|-\rangle|-\rangle + |-\rangle|-\rangle|+\rangle|+\rangle - (|+\rangle|-\rangle + |-\rangle|+\rangle)(|+\rangle|-\rangle + |-\rangle|+\rangle)$. Similary, there are $2^{n-1}$ such states that can be made with $2n$ spins subject to the constraint that no subset of the spins can form a conglomerate with spin greater than 1. A more sophisticated analysis shows that all $k > 1$ are effectively in the large-$k$ limit [7].

These braiding eigenvalues can be derived in an appealing way from the Chern-Simons theory. This approach begins with the observation [7] that the functional integral of our effective field theory with Wilson loop insertions in the spin-1/2 representation is equal to the Jones polynomial of the loops evaluated at $q = e^{\pi i/4}$.

$$\int D\alpha \;\; \mathrm{Tr}_{1/2} \left\{ \mathcal{P} e^{i\oint_\gamma a} \right\} \;\; e^{\frac{2}{4\pi}\int d^3x\, \epsilon^{\mu\nu\lambda}\left(\alpha^a_\mu \partial_\nu \alpha^a_\lambda + \frac{2}{3} f_{abc}\alpha^a_\mu \alpha^b_\nu \alpha^c_\lambda\right)} \;\; = \;\; V_\gamma(e^{\pi i/4}) \tag{20}$$

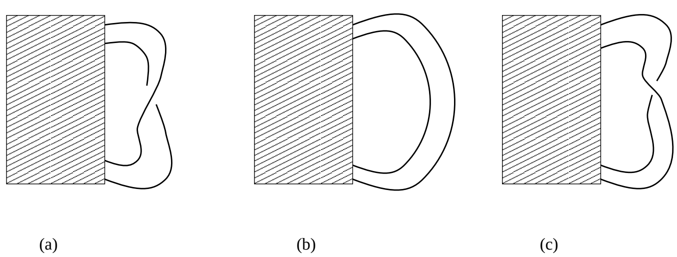


FIG. 1. The loops (a) $\gamma$ (b) $\gamma'$ and (c) $\gamma''$ which enter the skein relation. The three loops differ only by the braiding shown. The common shaded section is arbitrary.

For our purposes, the Wilson loops represent the world-lines of quasiparticles and the Jones polynomial will tell us how quantum-mechanical amplitudes depend on the braiding of the world-lines.

Let us recall a few facts about the Jones polynomial, $V_\gamma(q)$. It is a Laurent series in one variable, $q$, which is a topological invariant of a knot, $\gamma$. From the point of view of Chern-Simons theory, it is the next in a hierarchy of invariants whose first member is the linking number, which determines the acquired phase in Abelian statistics. It is defined by the statement that $V_\gamma(q) = 1$ if $\gamma$ is the unknot and by the skein relation:

$$q^{-1}\, V_\gamma(q) - q\, V_{\gamma''}(q) = \left(q^{1/2} - q^{-1/2}\right) V_{\gamma'}(q) \tag{21}$$

where $\gamma'$ and $\gamma''$ are obtained by performing successive counter-clockwise half-braids of any two world-lines in $\gamma$ as in figure 1.

Since (21) tells us how amplitudes are modified by braiding operations, it gives us a direct handle on the eigenvalues of the braiding matrix. Consider an arbitrary state $|\psi\rangle$ in the two-dimensional space of states with four quasiholes. According to the skein relation,

$$q^{-1}\,|\psi\rangle - q\,B^2|\psi\rangle = \left(q^{1/2} - q^{-1/2}\right) B|\psi\rangle \tag{22}$$

where $B$ is the braiding operator for two of the quasiparticles. (22) implies a quadratic equation for the eigenvalues of $B$ which yields the eigenvalues $e^{-3\pi i/8}$ and $e^{\pi i/8}$, again in agreement with (8).

We can also obtain the quasiparticle statistics from the braiding matrices of the spin-1/2 vertex operators in the $SU(2)_2$ WZW model. According to [9], these follow from the fusion rules and anomalous dimensions in that conformal field theory.

## IV. EFFECTIVE FIELD THEORY OF THE $\nu = 1/2$ PFAFFIAN STATE

We would now like to deform the theory (17) so that it describes the fermionic Pfaffian state at $\nu = 1/2$. (Of course, by the same procedure, we could also obtain the effective field theory for fermionic Pfaffian states at any even denominator or for bosonic Pfaffian states at any odd denominator.) The idea is to use the flux-attachment procedure [11–14] to change the bosons into fermions and simultaneously change the filling fraction from $\nu = 1$ to $\nu = 1/2$. To do this, we introduce a $U(1)$ gauge field, $c^\mu$, which couples to the charge current, $\epsilon^{\mu\alpha\beta}F^3_{\alpha\beta}$. The field $c^\mu$ will attach a flux tube to each electron through the Chern-Simons equation:

$$\epsilon^{\mu\alpha\beta}\partial_\alpha c_\beta = -j^\mu = \epsilon^{\mu\alpha\beta}F^3_{\alpha\beta} \tag{23}$$

This equation follows from the action

$$S = \frac{2}{4\pi}\int d^3x\,\epsilon^{\mu\nu\lambda}\left(a^a_\mu\partial_\nu a^a_\lambda + \frac{2}{3}f_{abc}a^a_\mu a^b_\nu a^c_\lambda\right) + \frac{1}{2\pi}c_\mu\epsilon^{\mu\alpha\beta}F^3_{\alpha\beta} - \frac{1}{4\pi}c_\mu\epsilon^{\mu\alpha\beta}\partial_\alpha c_\beta \tag{24}$$

which is our proposed effective field theory for the fermionic Pfaffian state at $\nu = 1/2$.

Equation (24) needs some explanation because the term which couples $a^a_\mu$ and $c_\mu$ breaks the $SU(2)$ gauge invariance down to the smaller $U(1)$ subgroup generated by $T_3$ (as does the coupling to the electromagnetic field, which has the same form). The action (17) and the equations of motion which follow from it are invariant under the transformation

$$a^a_\mu\,T_a \rightarrow g\,a^a_\mu\;T_a g^{-1} - \partial_\mu g\,g^{-1} \tag{25}$$

where $g$ is an $SU(2)$-valued function which must become identity at the boundary. As a result of (25), the time-evolution of $a^a_\mu$ (in the bulk) is not well-defined because a given set of initial conditions can lead to infinitely many possible solutions, each related to the others by the gauge transformation (25). To put it differently, the action does not specify a dynamics for the longitudinal part of the gauge field, which decouples from the transverse part. On the other hand, gauge-invariant quantities, such as $F^a_{\alpha\beta}F^a_{\mu\nu}$ have perfectly well-defined time-evolution since they are independent of the longitudinal part of $a^a_\mu$. For calculational purposes, we can choose a particular gauge, thereby specifying a dynamics for the longitudinal part of $a^a_\mu$. Of course, gauge-invariant quantities will not depend on this choice.

In (24), however, the $c_\mu\epsilon^{\mu\alpha\beta}F^3_{\alpha\beta}$ term breaks gauge-invariance by coupling the transverse part of the gauge field to the longitudinal part. As a result of its coupling to the longitudinal part of $a^a_\mu$, the transverse part no longer has a well-defined time-evolution either. We can only make sense of the action (24) if we understand the $SU(2)$ part of the action to be gauge-fixed. One possible choice is Coulomb gauge, $a^a_0 = 0$, or the weaker condition $a^1_0 = a^2_0 = 0$ which preserves the $U(1)$ gauge symmetry generated by $T_3$. In this gauge, there is a constraint,

$$F^a_{12} = 0 \tag{26}$$

which generates time-independent gauge transformations. We can impose another condition to eliminate this residual gauge symmetry, but we do not have to. Once we have imposed Coulomb gauge, a given set of initial conditions leads to a unique solution. The time-independent gauge transformations connect different gauge-equivalent initial conditions, but do not render the time evolution ill-defined. In other words, we have a hugely redundant, but completely well-defined dynamics. Hence, we can take (24) as our effective field theory, provided we understand the $SU(2)$ gauge field to be gauge-fixed in the Coulomb gauge or another suitable gauge.

Alternatively, and more profoundly, we can view the $c_\mu \epsilon^{\mu\alpha\beta} F^3_{\alpha\beta}$ term as a gauge-fixed form of the gauge invariant operator $\mathcal{L}_{\rm spin}$

$$\mathcal{L}_{\rm spin} \equiv \frac{1}{2\pi} \left( c_\mu + \partial_\mu \omega \right) \epsilon^{\mu\alpha\beta} F^a_{\alpha\beta} \phi^a \tag{27}$$

which is manifestly gauge invariant under both $U(1)$ and $SU(2)$ gauge transformations. The scalar field $\omega$ is chosen to transform under an arbitrary $U(1)$ gauge transformation $\alpha(x)$ as $\omega(x) \to \omega(x) - \alpha(x)$, and the scalar field $\phi^a$ transforms like a vector of the adjoint representation of the $SU(2)$ gauge group, and, as such, it transforms like $\phi^a(x) T^a \to g(x) \phi^a(x) T^a g^{-1}(x)$, where $g(x) \in SU(2)$. Then, after choosing the *unitary* gauge $\omega = 0$ and $\phi^a = (0,0,1)$, we recover gauge non-invariant term of (24).

Thus, we conclude that the effective field theory must contain a scalar field in the adjoint representation of $SU(2)$. The appearance of a scalar field in the adjoint representation of $SU(2)$ in the effective low-energy theory indicates that the microscopic quantum mechanical ground state must be characterized by a *condensate* or order parameter field which also transforms like a vector of the adjoint representation of $SU(2)$. Therefore, since this vector also breaks the global $SU(2)$ symmetry, the ground state must exhibit *spontaneous spin polarization.* Furthermore, since the scalar field $\omega$ breaks the $U(1)$ gauge symmetry, we must conclude that this effective field theory describes a state with off-diagonal long range order with a spin-triplet condensate. In other words, the effective field theory describes a state with a $p-wave$ condensate. This conclusion is in perfect agreement with the arguments of ref. [2].

Before leaving this subject, we should add one more improvement. Above, $\phi^a$ and $\omega$ have been introduced as purely formal objects. To treat them as proper physical variables, we should impose equations of motion corresponding to their variation. If we did that using (27) as it stands, we would obtain unsatisfactory constraint equations. We would also like to link these variables to more familiar order parameter fields. So we introduce two charged order parameter fields

$$\Delta_0 = |\Delta_0| e^{i\omega}, \quad \Delta^a = e^{i\omega} \phi^a \tag{28}$$

governed by the potential term

$$\mathcal{L}_{\rm pot} \;=\; - \lambda_1 (\mathbf{\Delta}\mathbf{\Delta}^+ - 1)^2 - \lambda_2 (\Delta_0 \Delta_0^+ - 1)^2 \tag{29}$$

with large parameters $\lambda_{1,2}$, thus fixing the amplitude of these fields. The Lagrangian density (27) then acquires a more familiar form:

$$\mathcal{L}_{\rm spin} = \frac{1}{4\pi} \Delta_0^+ \left( -i\partial_\mu + c_\mu \right) \Delta^a \epsilon^{\mu\alpha\beta} F^a_{\alpha\beta} + c.c. \tag{30}$$

Now we need no extra constraints. It is noteworthy that while the ordinary Chern-Simons terms are completely general covariant, potential term (29) is invariant only under volume preserving diffeomorphisms. This should not be disturbing, however, because there is a preferred density − though not a preferred shape − associated with incompressible quantum Hall liquid. The non-asymptotic form, with $\lambda_{1,2}$ finite, allows in principle for the description of localized vortex configurations.

Because the flux-attachment only affects the 'trivial' $U(1)$ part of the topological properties of the Pfaffian state, the quasiparticle statistics, degeneracy on the torus, *etc.* can be inferred from those at $\nu = 1$.

It is straightforward to generalize this construction to include more general filling factors. Essentially, all that is required is to attach an even number of flux quanta in addition to the procedure we used to map bosons to fermions. This is the standard procedure that is followed to generate the Jain fractions in the fermionic version of the abelian Chern-Simons theory of the FQHE [14]. There is, however, a significant subtlety concerning the global consistency of the implementation of flux attachment commonly used in the condensed matter literature, when it is applied to closed surfaces of non-trivial topology. Indeed, the action for the Abelian gauge field is ordinarily written with a Chern-Simons term with a coefficient $\theta = 1/(2\pi m)$, where $m$ is an even integer. This is inconsistent with the requirement of quantization of the Chern-Simons coupling constant needed for a Chern-Simons gauge theory on a closed manifold [15]. We should instead take this coefficient to be 1 and put the coefficient $1/\theta = (2\pi m)$ in front of the term (27) which couples the two gauge fields. In the Appendix we give a more careful discussion of this point, and derive the proper procedure from first principles.

## V. DISCUSSION

1. A main motivation for our effective field theory is to establish the existence of a universality class of quantum Hall states of which the Pfaffian state (3) is a representative. The most salient property of this universality class

is a $2^{n-1}$-fold degenerate set of $2n$-quasihole states which transform as the spinor representation of $SO(2n)$ as the quasiholes wind about each other. This is a bit worrisome since arbitrary perturbations might be expected to break this degeneracy. In particular, impurities or small variations in the inter-electron interactions could, potentially, do this, thereby spoiling the non-Abelian statistics. With the effective field theories (17) or (24) in hand, however, it is clear that this degeneracy is, indeed, stable in the long-wavelength limit. The leading perturbations are Maxwell terms of the form $F^{a}_{\mu\nu}F^{\mu\nu a}$ which are irrelevant by one power of $q$ or $\omega$ compared to the Chern-Simons terms. Perturbations which couple to the charge density, such as a random potential or Coulomb interactions, couple to the field strength of the $U(1)$ gauge field in (24). Hence, (24) is just as stable as an Abelian quantum Hall state, at least as far as such perturbations are concerned.

2. A Landau-Ginzburg theory of the Pfaffian state should take a paired order parameter as its starting point. Indeed, this is strongly suggested by the appearance of the fields $\Delta_0, \Delta^a$ in the gauge-invariant form of the effective field theory (30). This leads one to the Landau-Ginzburg theory of a superconductor, but with an Abelian Chern-Simons field to cancel the magnetic field or, in other words, to a theory almost identical to the theory of a Laughlin state (11). The crucial difference is that the paired order parameter must have a structure which allows the existence of neutral fermionic modes at vortex cores, where the gap to an unpaired fermion vanishes. The reader may recall that Bogoliubov-de Gennes quasiparticles in a superconductor become neutral fermions on the Fermi surface and are therefore natural candidates for the zero modes we are looking for. We could then, as in [4–6], interpret the $2^{n-1}$-fold degeneracy in terms of the occupation of fermionic zero modes associated with the vortices. This raises the interesting prospect that the duality transformation between the Landau-Ginzburg theory – in which the electrons are the fundamental objects – and the dual theory of equation (17) or (24) – in which the quasiparticles are fundamental – is a highly non-trivial transformation relating two theories which, at first glance, appear to be radically different.

3. The observed plateaus at $\nu = 5/2$ (in other words, $\nu = 1/2$ in the second Landau level) in a single-layer system and $\nu = 1/2$ in a double-layer system are promising hunting grounds for excitations with non-Abelian statistics. One way to experimentally determine whether either one is described by the Pfaffian state (2) is to measure the quasihole statistics. To do this, we would like to observe how the quantum state of the system transforms when quasiholes are braided. An elegant way to do this utilizes the two point-contact interferometer proposed by Chamon, *et. al.* [16].

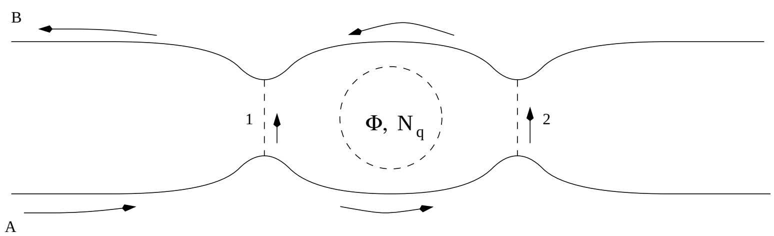


FIG. 2. In the two point-contact interferometer of Chamon, *et. al.*, quasiholes can tunnel from the lower edge to the upper edge by one of two interfering paths. The interference is controlled by varying the flux, $\Phi$, and number of quasiholes, $N_q$, in the central region.

In this device (see figure 2), quasiholes injected at point A along the bottom edge of the quantum Hall bar can tunnel to the other edge at either of two point-contacts. A quasihole which tunnels at the second point-contact will follow a path which encircles a central region containing flux $\Phi$ as well as $N_q$ quasiholes. As a result of the flux $\Phi$, it will aquire an Aharonov-Bohm phase which depends on its fractional charge. Its state will also be tranformed because its trajectory braids the $N_q$ quasiholes in the central region. Consequently, the interference between the two tunneling paths will depend on the charge and statistics of the quasiholes. As discussed by Chamon, *et. al.* [16], if we hold the electron number (and therefore the quasihole number) in the central region fixed, then the conductance will oscillate as a function of $\Phi$ with period $\frac{e}{e^*}\Phi_0$, where $e^*$ is the quasihole charge. If, on the other hand, we vary $N_q$, we can probe the statistics. Let us suppose that a quasihole which is injected at point $A$ on the bottom edge in figure 2 and tunnels at the first point-contact arrives at point B in state $|\psi\rangle$ and a quasihole which tunnels at the second point contact is in the state $e^{i\alpha}\, B_{N_q}|\psi\rangle$, where $B_{N_q}$ is the braiding operator for the quasihole to encircle the quasiholes in the central region and $e^{i\alpha}$ is the additional Aharonov-Bohm and dynamical phase aquired along the second path. Then the current which is measured at $B$ will be proportional to

$$\frac{1}{2}\left(|t_1|^2 + |t_2|^2\right) \,+\, \mathrm{Re}\left\{t_1^* t_2\, e^{i\alpha}\, \langle\psi|B_{N_q}|\psi\rangle\right\} \tag{31}$$

where $t_1$ and $t_2$ are the tunneling amplitudes at the two point-contacts. Now, the matrix element $\langle\psi|B_{N_q}|\psi\rangle$ is given precisely by the expectation value in the effective field theory (17) of the Wilson lines of figure 3 or, simply, by the

Jones polynomial $V_{N_q}(e^{\pi i/4})$ of these loops.

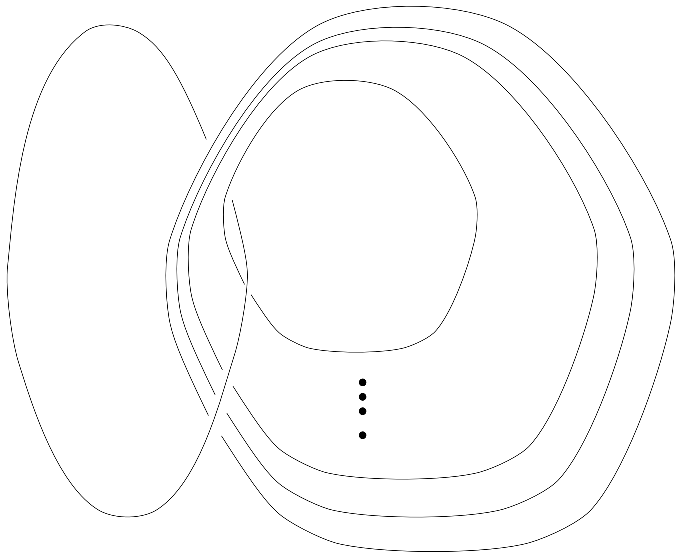

FIG. 3. The matrix element $\langle\psi|B_{N_q}|\psi\rangle$ is the expectation value of these Wilson lines. The loop on the left represents a quasihole which tunnels at the second point contact, thereby encircling the $N_q$ quasiholes on the right.

In other words, if the Pfaffian state exists in nature, a two point-contact interferometer will measure the Jones polynomial! By studying the dependence on $N_q$ – the conductance will exhibit the periodicities of the two eigenvalues of $B_{N_q}$ – we will be able to extract the non-Abelian statistics.

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## APPENDIX A:

We will show how to define flux attachment in a manner compatible with the requirement of quantization of the abelian Chern-Simons coupling constant or, what is the same, of invariance under large gauge tranformations. This issue does not arise for the non-abelian sector since its coupling constant is already correctly quantized.

Consider a theory of particles (in first quantization) which interact with each other as they evolve in time. We will assume in what follows that the particles are fermions (in two spacial dimensions) and that their worldlines never cross. The actual choice of statistics is not important in what follows but the requirement of no crossing is important and, for bosons, it implies the assumption that there is a hard-core interaction while for fermions the Pauli principle takes care of this issue automatically. For simplicity, we will assume that the time evolution is periodic, with a very long period.

The worldlines of the particles can be represented by a conserved current $j_\mu$. For a given history of the system, the worldlines form a braid with a well defined linking number $\nu_L[j_\mu]$, given by

$$\nu_L[j_\mu] = \int d^3x \; j_\mu(x) B^\mu(x) \tag{A1}$$

where $j_\mu$ and $B_\mu$ are related through Ampère's Law

$$\epsilon_{\mu\nu\lambda}\partial^\nu B^\lambda(x) = j_\mu(x) \tag{A2}$$

Under the assumption of the absence of crossing of the worldlines of the particles, the linking $\nu_L[j_\mu]$ is a topological invariant.

Thus, if $S[j_\mu]$ is the action for a given history, the quantum mechanical amplitudes of all physical observables remain unchanged if the action is modified by

$$S[j_\mu] \to S[j_\mu] + 2\pi n \nu_L[j_\mu] \tag{A3}$$

where $n$ is an arbitrary integer.

The quantum mechanical amplitudes are sums over histories of the particles, and take the form

$$\text{Amplitude} \propto \sum_{[j_\mu]} e^{iS[j_\mu]+2\pi i n \nu_L[j_\mu]} e^{i\phi[j_\mu]} \tag{A4}$$

where $\phi[j_\mu]$ is a phase factor which accounts for the statistics of the particles.

However, the amplitudes remain unchanged if the integrand of Eq. A4 is mulitiplied by 1 written as the expression

$$1 \equiv \int \mathcal{D}b_\mu \prod_x \delta(\epsilon_{\mu\nu\lambda}\partial^\nu b^\lambda - j_\mu) = \mathcal{N} \int \mathcal{D}b_\mu\ \mathcal{D}a_\mu\ \exp\left(\frac{i}{2\pi}\int d^3x\ a^\mu\left[\epsilon_{\mu\nu\lambda}\partial^\nu b^\lambda - j_\mu\right]\right) \tag{A5}$$

where $\mathcal{N}$ is a normalization constant and we have used a representation of the delta function in terms of a Lagrange multiplier vector field $a_\mu$. Notice that, since $j_\mu$ is locally conserved (*i. e.* $\partial_\mu j^\mu = 0$), these expressions are invariant under the gauge transformations $a_\mu(x) \to a_\mu(x) + \partial_\mu \Lambda(x)$.

After using the constraint $j = \partial \wedge b$, the amplitude can also be written in the equivalent form

$$\text{Amplitude} \propto \sum_{[j_\mu]} \int \mathcal{D}b_\mu\ \mathcal{D}a_\mu\ e^{iS[j_\mu]+2\pi i n \nu_L[j_\mu]} e^{i\phi[j_\mu]} e^{i\int d^3x a^\mu(x) \frac{1}{2\pi}\left[\epsilon_{\mu\nu\lambda}\partial^\nu b^\lambda - j_\mu\right]} \tag{A6}$$

We can then compute this amplitude as a path integral of a theory in which the particles whose worldlines are represnted by the currents $j_\mu$, interact with the gauge fields $a_\mu$ and $b_\mu$. These interactions are encoded in the effective action

$$S_{\text{eff}}[a,b,j] = \frac{1}{2\pi} a^\mu\left(\epsilon_{\mu\nu\lambda}\partial^\nu b^\lambda - j_\mu\right) + \frac{2n}{4\pi}\epsilon_{\mu\nu\lambda} b^\mu \partial^\nu b^\lambda \tag{A7}$$

where we have solved the constraint $j = \partial \wedge b$ to write the term of the winding number in the form of a Chern-Simons action for the gauge field $b_\mu$ [17,18]. Hence, the amplitudes can be written in terms of a path integral over an abelian Chern-Simons gauge field with a correctly quantized coupling constant equal to $\frac{2n}{4\pi}$.

The usual form of the flux-attachment transformation is found by integrating out the gauge field $b$. For vanishing boundary conditions at infinity, this leads to an effective action for the field $a_\mu$ of the conventional form [14]

$$S_{\text{eff}}[a] = \frac{1}{2}\frac{1}{2\pi 2n}\int d^3x \epsilon_{\mu\nu\lambda} a^\mu \partial^\nu a^\lambda \tag{A8}$$

This form of the effective action is not valid for manifolds with non-trivial topology. However, Eq. A7 is correct in all cases as it is both invariant under both local and large gauge transformations. Notice that the gauge field $b_\mu$ is the dual field referred to in the rest of the paper and it plays a central role in Wen's construction of the Abelian FQH hierarchy [19].

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