

# CHERN-SIMONS QUANTUM FIELD REPRESENTATION OF PLANAR FERROMAGNETS

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*We represent the two - dimensional planar classical continuous Heisenberg spin model as a constrained Chern-Simons gauged nonlinear Schrödinger system. The hamiltonian structure of the model is studied, allowing the quantization of the theory by the gauge invariant approach. A preliminary study of the quantum states is displayed and several physical consequences in terms of anyons are discussed.*

**1.** Chern-Simons (CS) gauged models have recently attracted a great attention in the description of several phenomena in condensed matter physics as the Fractional Quantum Hall Effect (FQHE) and the High Temperature Superconductivity [1, 2]. In this context, models of the Landau-Ginzburg type have been proposed as effective field-theories [3, 4]. In the stationary limit, soliton solutions to the self-dual reductions of these models appear. These can be exploited to represent Laughlin's quasiparticles and anyons [5]. On the other hand, the present authors showed that the self-dual CS system can be obtained, in the tangent space representation, from the stationary two-dimensional Landau - Lifshitz equation (LLE) [6]. This is the classical continuous isotropic version of the Heisenberg model, obtained by the discrete quantum version in the framework of the coherent state representation [7]. In particular, soliton solutions of the former system turn out to be related to magnetic vortices of the LLE. More generally, the LLE in 2+1 dimensions takes the form of a constrained nonlinear Schrödinger system (cNLS) for two charged matter fields, coupled to a CS gauge potential [8]. This correspondence suggests an explanation to the phenomenological analogy between the behaviour of magnetic vortices [9] and Hall particles in an external magnetic field [10].

Sec. 2 is devoted to a brief review of the Landau - Lifshitz model in 2+1

dimensions and its tangent space formulation, that is the CS gauged cNLS system. Furthermore, the conservation of some associated physical quantities is discussed. In Sec. 3 we study the lagrangian formulation and the hamiltonian structure of the CS gauged cNLS system. Finally, in Sec. 4 we apply the method of gauge invariant quantization to the above mentioned system.

**2.** In 2+1 dimensions the LLE for the local magnetization  $\vec{S}$  reads

$$\vec{S}_t = \vec{S} \times \nabla^2 \vec{S}, \quad (1)$$

where  $\vec{S}$  belongs to the 2-dimensional sphere  $\mathcal{S}^2$ . In terms of the stereographic projection

$$S_+ = S_1 + i S_2 = 2 \frac{\zeta}{1 + |\zeta|^2}, \quad S_3 = \frac{1 - |\zeta|^2}{1 + |\zeta|^2} \quad (2)$$

Eq. (1) becomes

$$\frac{1}{4} i \zeta_t + \zeta_z \bar{z} = 2 \frac{\zeta_z \zeta_{\bar{z}}}{1 + |\zeta|^2} \bar{\zeta} \quad (z = x_1 + i x_2, \bar{z} = x_1 - i x_2). \quad (3)$$

With ferromagnetic boundary conditions, that is  $\vec{S}(\mathbf{r}, t) \rightarrow (0, 0, -1)$  for  $|\mathbf{r}| \rightarrow \infty$ , the  $(x_1, x_2)$  plane is compactified. Thus  $\vec{S}$  defines maps of  $\mathcal{S}^2 \rightarrow \mathcal{S}^2$ , which are partitioned into homotopy classes, labeled by the integer-valued topological charge

$$Q = \frac{1}{4\pi} \int \vec{S} \cdot (\partial_1 \vec{S} \times \partial_2 \vec{S}) d^2x = \frac{1}{\pi} \int \frac{|\zeta_{\bar{z}}|^2 - |\zeta_z|^2}{(1 + |\zeta|^2)^2} d^2x. \quad (4)$$

The energy of the system takes the form

$$W = \frac{1}{2} \int (\nabla \vec{S})^2 d^2x = 4 \int \frac{|\zeta_{\bar{z}}|^2 + |\zeta_z|^2}{(1 + |\zeta|^2)^2} d^2x, \quad (5)$$

which compared with Eq. (4) leads to the Bogomol'nyi inequality  $W \geq 4\pi |Q|$ . When the equality holds, the field  $\zeta$  has to be an (anti)holomorphic function of  $z$ , leading to the Belavin-Polyakov instantons [11].

In our approach the spin  $\vec{S}$  is represented by the matrix  $S = S_\alpha \sigma_\alpha$  ( $\alpha = 1, 2, 3$ ), where  $\sigma_\alpha$  are the Pauli matrices. Then, we diagonalize  $S$  by resorting to a right-invariant local  $U(1)$  transformation  $g$ , such that  $S = g \sigma_3 g^{-1}$ . The matrix  $g$  belongs to  $U(2)$  and has the form

$$g = \pm [2(1 + S_3)]^{-1/2} \begin{pmatrix} 1 + S_3 & -\bar{S}_+ \\ S_+ & 1 + S_3 \end{pmatrix} \exp\left(i\sigma_3 \frac{\eta}{2}\right), \quad (6)$$

where  $\eta$  is an arbitrary real function. The corresponding chiral current  $J_\mu$  ( $\mu = 0, 1, 2$ ) is decomposable into a diagonal and an off-diagonal part:

$$J_\mu = g^{-1} \partial_\mu g = \left( \frac{i}{4} \right) \sigma_3 V_\mu + \begin{pmatrix} 0 & -\bar{q}_\mu \\ q_\mu & 0 \end{pmatrix}. \quad (7)$$

By definition, the chiral current  $J_\mu$  satisfies the zero curvature condition

$$\partial_\mu J_\nu - \partial_\nu J_\mu + [J_\mu, J_\nu] = 0. \quad (8)$$

Its off-diagonal and diagonal parts are

$$D_\mu q_\nu = D_\nu q_\mu, \quad (9)$$

$$[D_\mu, D_\nu] = -\frac{i}{2} (\partial_\mu V_\nu - \partial_\nu V_\mu) = -2 (\bar{q}_\mu q_\nu - q_\mu \bar{q}_\nu), \quad (10)$$

where we have introduced the covariant derivative  $D_\mu = \partial_\mu - \frac{i}{2} V_\mu$ .

In the variables  $q_\mu$ , the equation of motion (1) becomes

$$q_0 = i D_m q_m, \quad (11)$$

which allows us to eliminate the function  $q_0$  from Eqs. (10-11). Now, let us introduce the new fields

$$\psi_\pm = \frac{(q_1 \pm i q_2)}{2\sqrt{\pi}}, \quad A_0 = V_0 - 8(|\psi_+|^2 + |\psi_-|^2), \quad A_i = V_i \quad (i = 1, 2) \quad (12)$$

and, correspondingly, the new covariant derivatives  $D_\mu = \partial_\mu - (i/2) A_\mu$ . Then we notice that the expressions of  $\psi_\pm$  and  $A_j$  in terms of  $\zeta$  are

$$\psi_\pm = \frac{1}{2\sqrt{\pi}} \frac{(\partial_1 \pm i \partial_2) \zeta}{(1 + |\zeta|^2)} e^{i\eta}, \quad A_j = 2i \frac{(\zeta \partial_j \bar{\zeta} - \bar{\zeta} \partial_j \zeta)}{(1 + |\zeta|^2)} + 2\partial_j \eta. \quad (13)$$

For  $\mu = 1$  and  $\nu = 2$ , we see that Eq. (9) takes the form

$$\gamma \equiv (D_1 + i D_2) \psi_- - (D_1 - i D_2) \psi_+ = 0. \quad (14)$$

This equation can be interpreted as a geometrical constraint for the CS gauged NLS system, arising from the remaining equations (9) and Eq. (10), i.e.

$$i D_0 \psi_\pm + (D_1^2 + D_2^2) \psi_\pm + 8\pi |\psi_\pm|^2 \psi_\pm = 0, \quad (15.a)$$

$$\partial_2 A_1 - \partial_1 A_2 = 8\pi (J_0^+ - J_0^-), \quad (15.b)$$

$$\partial_0 A_j - \partial_j A_0 = -8\pi \varepsilon_{j\ell} (J_l^+ - J_l^-), \quad (15.c)$$

where

$$J_0^\pm = |\psi_\pm|^2, \quad J_l^\pm = i(\psi_\pm \bar{D}_l \bar{\psi}_\pm - \bar{\psi}_\pm D_l \psi_\pm) \quad (l = 1, 2). \quad (16)$$

Here we introduce the charge density currents satisfying the continuity equation  $\partial_\mu J_\mu^\pm = 0$ . Then, the matter fields  $\psi_\pm$  carry a conserved electric charge and we can interpret the quantity

$$\mathcal{Q} = \int (J_0^+ - J_0^-) d^2x \quad (17)$$

as the (conserved) total electric charge. In terms of  $\psi_\pm$  it is easy to see that  $Q$  and  $\mathcal{Q}$  coincide, thus we are led to the quantization of the total electric charge. Furthermore, Eq. (15.b) implies the quantization of the magnetic flux:  $\Phi = \int B d^2x = \int (\partial_1 A_2 - \partial_2 A_1) d^2x = -8\pi Q$ .

Moreover, the continuity equations for the currents  $J_\mu^\pm$  predict that the total number of particles (anyons)

$$\mathcal{N} = \int (J_0^+ + J_0^-) d^2x \quad (18)$$

is also a conserved quantity. Now, comparing Eq. (5) with (18), we obtain  $W = 4\pi\mathcal{N}$ . Then, in the tangent space formulation, the Bogomol'nyi inequality takes the form  $|Q| \leq \mathcal{N}$ . Of course, the equality holds when  $\psi_+$ , or  $\psi_-$ , vanishes. In such a case, the system (14 - 15) reduces to the self-dual CS model [2]. This model is completely determined in terms of the solutions of the Liouville equation. This simple observation enables us to identify the magnetic vortices, or equivalently the instantons of Belavin-Polyakov type, with the solitons of the self-dual CS model, that is with the anyons. In particular, from (17) we have an explicit correspondence between the magnetic topological charge and the anyon electric charge, whose quantization finds here a geometrical interpretation. In ref. [8] the more general static case  $(D_1 + iD_2) \psi_- = (D_1 - iD_2) \psi_+ = 0$  is treated. This leads to a special reduction of the euclidean conformal affine Toda system, which contains the sinh-Gordon, or the Poisson-Boltzmann equation, in a particular gauge.

Finally, we notice that the constraint (14) is identically satisfied using the expressions (13). Indeed, we find the identity

$$(D_1 + iD_2) \psi_- = (D_1 - iD_2) \psi_+ = \frac{1}{2} \left[ \zeta_z \bar{z} - 2 \frac{\zeta_z \zeta_{\bar{z}} \bar{z}}{1 + |\zeta|^2} \bar{\zeta} \right] \frac{e^{i\eta}}{1 + |\zeta|^2}. \quad (19)$$

Equating to zero the quantity in the square brackets, we recognize the stationary LLE in the stereographic variable (see Eq. (3)). Thus, all static solutions to Eq. (1) are mapped, modulo gauge transformations, into solutions to the sinh-Gordon reduction studied in ref. [8].

**3.** System (15) can be derived from the Lagrangian

$$\mathcal{L} = \frac{i}{2} (\bar{\psi} D_0 \psi - \bar{D}_0 \bar{\psi} \psi) - \bar{D}_a \bar{\psi} D_a \psi + 4\pi (\bar{\psi} \sigma_3 \psi) (\bar{\psi} \psi) + \frac{1}{32\pi} \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda, \quad (20)$$

where we have used the spinor notation  $\psi^T = (\psi_+, \psi_-)$ , with  $\bar{\psi} = \psi^+ \sigma_3$ , and the summation over  $a = 1, 2$  is understood. Let us observe that the strength of the self-interaction and the CS coupling are fixed by the original spin model. The specific ratio between these constants allows us to perform the self-dual reduction, which is a very special subcase in more general and the phenomenological theories of CS type [2]. Moreover, in the effective theories for FQHE the CS coupling is related to the so-called *filling factor*  $\nu$ , which in the present case is  $1/2$ , where the units  $\hbar = c = 1$  and  $e = 1/2$  have been used. This result is consistent with the so-called chiral spin fluid model [1 - 12]. Moreover, recent experimental observations in double-layer samples indicate the compatibility of  $\nu$ , with values which are inverse of even numbers [13].

A final remark concerns the parity invariance of the Lagrangian (20), which changes the total sign under the transformation

$$\begin{aligned} x_1 &\rightarrow x_1, & x_2 &\rightarrow -x_2 \\ \psi_+ &\leftrightarrow \psi_-, & D_1 + iD_2 &\leftrightarrow D_1 - iD_2. \end{aligned} \quad (21)$$

So that equations of motion and the constraint (14) are both left invariant. This fact is an obvious consequence of the parity invariance of the spin model (1) and it is another example, against the common belief, that the presence of the CS interaction breaks the P and T invariance.

Now let us enter into the detail of the hamiltonian structure of the model (14 - 15) by resorting to the first order Lagrangian method proposed in ref. [14], which implements the classical Dirac method for constrained Hamiltonian systems [15].

First, we consider as dynamical variables  $\xi = (\psi_+, \bar{\psi}_+, \psi_-, \bar{\psi}_-, A_1, A_2)$  and, defining  $a = (\frac{i}{2} \bar{\psi}_+, -\frac{i}{2} \psi_+, -\frac{i}{2} \bar{\psi}_-, \frac{i}{2} \psi_-, \frac{1}{32\pi} A_2, -\frac{1}{32\pi} A_1)$ , we rearrange (20) in the form

$$\mathcal{L}' = a_m \dot{\xi}^m - \mathcal{H}(\xi) + \frac{1}{2} A_0 \Gamma_1, \quad (22)$$

$$\mathcal{H} = \bar{D}_a \bar{\psi} D_a \psi - 4\pi (\bar{\psi} \sigma_3 \psi) (\bar{\psi} \psi) \quad (23)$$

$$\Gamma_1 \equiv \frac{\delta \mathcal{L}}{\delta A_0} = \bar{\psi} \psi + \frac{1}{8\pi} \epsilon^{ij} \partial_i A_j = 0. \quad (24)$$

Thus  $\mathcal{L}'$  describes the system (15 a,c) constrained by Eq. (24), that is the Chern-Simons version of the Gauss law (15.b). In (22)  $A_0$  plays the role of Lagrange multiplier. Furthermore,  $\mathcal{H}$  would be the density of the Hamiltonian  $H$ . In fact, the corresponding Euler-Lagrange equations take the hamiltonian form

$$\dot{\xi}^i = f_{ij}^{-1} \frac{\partial \mathcal{H}}{\partial \xi^j} = \{ \xi^i, H \}, \quad (25)$$

where  $f_{ml} = \partial a_l / \partial \xi^m - \partial a_m / \partial \xi^l$  is a nonsingular symplectic matrix and we have defined the Poisson brackets by  $\{ \xi^m, \xi^l \} = f_{ml}^{-1} \delta^2(\mathbf{x} - \mathbf{y})$ . Specifically, in our case the nonvanishing Poisson brackets are

$$\{ \psi_{\pm}(\mathbf{x}), \bar{\psi}_{\pm}(\mathbf{y}) \} = \mp i \delta^2(\mathbf{x} - \mathbf{y}), \quad \{ A_i(\mathbf{x}), A_j(\mathbf{y}) \} = 16\pi \epsilon_{ij} \delta^2(\mathbf{x} - \mathbf{y}). \quad (26)$$

By using Eq. (33), one easily shows that  $\dot{\Gamma}_1 = \{\Gamma_1, H\} = 0$ ; consequently, in phase space the Gauss' law defines an invariant submanifold under the dynamics. On the other hand, Eq. (14) provides the further complex primary constraint  $\gamma = 0$ , whose real and immaginary parts will be denoted by  $\Gamma_2$  and  $\Gamma_3$ . By evaluating the Poisson brackets, we find that  $\{\gamma, H\} \approx 0$ , i.e.  $\dot{\gamma}$  is vanishing only on the submanifold defined by (14) or, in other words,  $\Gamma_1$  and  $\Gamma_2$  are weakly invariant under time evolution. Furthermore, they do not produce secondary constraints. However, since  $\{\Gamma_1, \Gamma_2, \Gamma_3\}$  is a set of first class constraints (i.e.,  $\{\Gamma_i, H\} \approx 0$  and  $\det(\{\Gamma_i, \Gamma_j\}) \approx 0$ ), we need a gauge-fixing condition, for example the Coulomb gauge condition  $\Gamma_4 = \partial_i A^i \approx 0$ , which is the most used in the litterature [16, 17, 18]. So, we are able to compute the Dirac brackets, defined by  $\{\xi^m, \xi^l\}_D \equiv \{\xi^m, \xi^l\} - \{\xi^m, \Gamma_s\}(d^{-1})_{ss'}\{\Gamma_{s'}, \xi^l\}$ , where  $d_{ss'} = \{\Gamma_s, \Gamma_{s'}\}$ . In our case, the non-vanishing matrix elements  $d_{ss'}$  are distributions in terms of  $\delta^2(\mathbf{x} - \mathbf{y})$  and its derivatives. Then, inversion can be performed only in implicit form. In order to avoid this type of problem, one can look for a different gauge-fixing condition. For instance, a more general non-local gauge fixing can be choosen [19], or the so-called "axial gauge" [20]. Nevertheless, in all generality the corresponding Dirac brackets are rational in the field variables. Then a quantization of the model seems to be rather difficult, because of the problem arising in the ordering and the inversion of the operators involved. This is one of the main reasons which address us to look for an alternative approach, which is invariant under the choice of the gauge.

4. In this Section, we will adopt a slightly different point of view, in order to tackle the above mentioned problems. Moreover, we would like to understand the quantum meaning of the geometrical constraints  $\gamma$ , as seen separately from the basic CS-NLS field theory. Then, a possible route consists in quantizing system (14 - 15) in two steps:

1. first, we apply the procedure of the gauge-invariant quantization to system (15) following the approach suggested in ref. [21];
2. then, we use the constraint  $\gamma$  to select the physical states among all the gauge invariant quantum states.

First of all, let us change the hamiltonian description by taking

$$\mathcal{H}' = \mathcal{H}(\xi) + f_0 \Gamma_0 + \frac{1}{2} A_0 \Gamma_1, \quad (27)$$

where we have as primary constraint  $\Gamma_0 \equiv \pi_0 = 0$ ,  $\pi_0$  being the momentum conjugated to the canonical variable  $A_0$ , such that  $\{A_0, \pi_0\} = \delta^2(\mathbf{x} - \mathbf{y})$ . This constraint is implicit in the previous formulation, since  $\partial \mathcal{L} / \partial \dot{A}_0 = 0$ .  $\Gamma_1$  is now a secondary constraint, arising from the Poisson bracket  $\{\pi_0, H\}$ . Finally, in the expression (27)  $f_0$  is an arbitrary function, which defines the evolution of  $A_0$ . Furthermore, noticing that the special canonical pair  $(A_1, A_2)$  breaks the rotational covariance of the theory, it is more convenient to consider the decomposition of the vector potential into a longitudinal and a transverse part  $A_i(\mathbf{x}) = A_i^L + A_i^T = 2\partial_i \eta(\mathbf{x}) + \epsilon_{ij} \partial_j \chi(\mathbf{x})$ . Recalling that the magnetic field is  $B = \epsilon^{ij} \partial_i A_j = -\partial^2 \chi$ , we derive formally the useful relation

$$A_i^T(\mathbf{x}) = -\epsilon_{ij} (\partial_j^{-1} B)(\mathbf{x}), \quad (28)$$

where we have defined  $\partial_j^{-1} f(\mathbf{x}) = \frac{1}{2\pi} \partial_j^{(x)} \int \ln |\mathbf{x} - \mathbf{y}| f(\mathbf{y}) d^2 y$ . Analogously, it is convenient to express  $\psi_{\pm}$  in terms of the canonical variables  $(P_{\pm}, Q_{\pm})$  by

$$\psi_{\pm} = \frac{1}{\sqrt{2}} (P_{\pm} \mp iQ_{\pm}). \quad (29)$$

Hence, the Poisson structure associated with the hamiltonian system given by (27) is

$$\begin{aligned} \{Q_{\pm}(\mathbf{x}), P_{\pm}(\mathbf{y})\} &= \delta^2(\mathbf{x} - \mathbf{y}), \quad \{A_0(\mathbf{x}), \pi_0(\mathbf{y})\} = \delta^2(\mathbf{x} - \mathbf{y}), \\ \{B(\mathbf{y}), \eta(\mathbf{x})\} &= 8\pi \delta^2(\mathbf{x} - \mathbf{y}). \end{aligned} \quad (30)$$

Furthermore, the equations of motion are supplemented by the first class constraints  $\Gamma_0 \approx \Gamma_1 \approx 0$ .

Now, we are look for a suitable canonical transformation, such that some of the new momenta are equal to the constraints. In such a way, the coordinates canonically conjugated to these momenta have an arbitrary time evolution and are remnants of the gauge invariance of the theory. The remaining canonical variables define the prominent dynamical degrees of freedom of the theory.

First, let us denote by  $(Q_{\pm}^*, P_{\pm}^*), (A_0^*, \pi_0^*), (\eta^*, \pi_1^*)$  new canonically conjugated coordinates, where the momenta  $\pi_i^*$  have to be set equal to the constraints  $\Gamma_i$ . This can be done through a suitable generating function  $F = F(Q_{\pm}, A_0, \eta; P_{\pm}^*, \pi_0^*, \pi_1^*)$ , satisfying the set of generalized Hamilton-Jacobi equations

$$\pi_i^* = \Gamma_i \left( Q_{\pm}, A_0, \eta; \frac{\partial F}{\partial Q_{\pm}}, \frac{\partial F}{\partial A_0}, \frac{\partial F}{\partial \eta} \right). \quad (31)$$

The integrability of this system is assured by  $\{\Gamma_0, \Gamma_1\} = 0$ , moreover its general solution is given by

$$\begin{aligned} F = \pi_0^* A_0 + \left( \pi_1^* + \frac{1}{2} P_+^{*2} - \frac{1}{2} P_-^{*2} \right) \eta + \\ \int \sqrt{P_+^{*2} - Q_+^2} dQ_+ + \int \sqrt{P_-^{*2} - Q_-^2} dQ_-. \end{aligned} \quad (32)$$

The corresponding canonical transformation reads

$$\begin{cases} \pi_0 = \pi_0^*, \quad A_0 = A_0^*, \quad -\frac{1}{8\pi} B = \pi_1^* + \frac{1}{2} P_+^{*2} - \frac{1}{2} P_-^{*2}, \quad \eta = \eta^*, \\ Q_{\pm} = P_{\pm}^* \sin \left( \frac{Q_{\pm}^*}{P_{\pm}^*} \mp \eta^* \right), \quad P_{\pm} = P_{\pm}^* \cos \left( \frac{Q_{\pm}^*}{P_{\pm}^*} \mp \eta^* \right). \end{cases} \quad (33)$$

Finally, the new gauge-invariant degrees of freedom are

$$\Phi_{\pm} = \frac{1}{\sqrt{2}} P_{\pm}^* \exp \left( \mp i \frac{Q_{\pm}^*}{P_{\pm}^*} \right) = \psi_{\pm} \exp(-i\eta), \quad \pi_{\pm}^* = \bar{\Phi}_{\pm}, \quad (34)$$

which fulfill the canonical brackets

$$\{\Phi_{\pm}(\mathbf{x}), \bar{\Phi}_{\pm}(\mathbf{y})\} = \mp i \delta^2(\mathbf{x} - \mathbf{y}). \quad (35)$$

In terms of these new variables, the Hamiltonian density becomes

$$\begin{aligned} \mathcal{H} = & (\partial_a + \frac{i}{2}A_a^T)\bar{\Phi}_+(\partial_a - \frac{i}{2}A_a^T)\Phi_+ - (\partial_a + \frac{i}{2}A_a^T)\bar{\Phi}_-(\partial_a - \frac{i}{2}A_a^T)\Phi_- \\ & - 4\pi (|\Phi_+|^4 - |\Phi_-|^4) + f^0\pi_0^* + \frac{1}{2}A_0^*\pi_1^*, \end{aligned} \quad (36)$$

where  $A_a^T(\mathbf{x}) = 8\pi \epsilon_{ab} \partial_b^{-1}(|\Phi_+|^2 - |\Phi_-|^2)(\mathbf{x})$ . Furthermore, the constraint  $\gamma$  takes the form

$$\gamma = (\partial_+ - \frac{i}{2}A_+^T)\Phi_- - (\partial_- - \frac{i}{2}A_-^T)\Phi_+, \quad (37)$$

where  $\partial_{\pm} = \partial_1 \pm i\partial_2$  and  $A_{\pm}^T = A_1^T \pm iA_2^T$ .

Now, let us observe that the relations (13) are algebraically invertible in terms of gauge invariant quantities when the topological density is non vanishing, namely

$$\zeta = \frac{i}{4\sqrt{\pi}} \frac{\Phi_- A_+^T - \Phi_+ A_-^T}{|\Phi_+|^2 - |\Phi_-|^2}. \quad (38)$$

Then, we can directly reconstruct the spin field in terms of  $\Phi_{\pm}$ , although it turns out to be a non local field in view of the expressions for  $A_{\pm}^T$ . In this regard, let us observe that, in the one-dimensional completely integrable case, the relation between  $\zeta$  and the NLS matter field is also analytic in nature, but we need to solve the associated linear spectral problem.

Furthermore, keeping in mind that the current densities (16) are manifestly gauge invariant, we can give the expressions of the linear and angular momentum for the model (14 - 15) in a gauge invariant form.

Firstly, in opposition to what happens for the electric charge densities, the linear momenta  $P_m^{\pm} = \int J_m^{\pm} d^2x$ , associated with the  $\Phi_{\pm}$  fields separately, are no longer conserved, but they evolve according to Newton's law

$$\frac{d}{dt}P_m^{\pm} = 8 \epsilon_{mn} \int (J_0^- J_n^+ - J_0^+ J_n^-) d^2x. \quad (39)$$

We notice that both  $\Phi_{\pm}$  are subjected to the same force. Therefore, the total kinetic momentum

$$P_m = P_m^+ - P_m^- = \int (\bar{\Phi}_+ \partial_m \Phi_+ - \bar{\Phi}_- \partial_m \Phi_-) d^2x + c.c. \quad (40)$$

is conserved, the " - " comes from the "negative mass" associated to  $\Phi_-$ . The total angular momentum  $M = M_+ - M_-$  is also conserved, with

$$M_{\pm} = \int (x_1 J_2^{\pm} - x_2 J_1^{\pm}) d^2x = \ell_{\pm} + s_{\pm} \quad (41)$$

and

$$\begin{aligned} l_{\pm} &= -i \int \bar{\Phi}_{\pm} (x_1 \partial_2 - x_2 \partial_1) \Phi_{\pm} d^2x + c.c., \\ s_{\pm} &= \int |\Phi_{\pm}|^2 (A_1^T x_2 - A_2^T x_1) d^2x. \end{aligned} \quad (42)$$

The quantities  $l = l_+ - l_-$  and  $s = s_+ - s_-$  are interpreted as orbital and spin angular momentum, respectively. In particular, the general result  $s = 2Q^2$  suggests that vortices have non zero momentum also in the static case.

Finally, since the magnetization  $\vec{\mathcal{M}} = \int \vec{S} d^2x$  is a conserved non-local quantity for the LLE system, via Eqs. (38) and (2) we find a further gauge invariant conserved quantity for the system (14 - 15), related to the original global  $SU(2)$  symmetry.

We are now ready to pass on the quantum theory in a straightforward manner, just by replacing the canonical brackets by  $i$  times the commutators (or the anti-commutators, if the matter fields are supposed to be fermionic). At the quantum level, the first class constraints  $\Gamma_i$  become the operators  $\hat{\Gamma}_i$ , which must annihilate the physical states. This fact implies that the quantum states are independent of  $A_0$  and invariant under time-independent gauge-transformations. Therefore, the gauge - invariant operators of the theory must commute with  $\hat{\Gamma}_i$ . Then, recalling the Poisson brackets for the Hamiltonian system described by (36), we can establish the correspondence  $\Phi_{\pm} \rightarrow \hat{\Phi}_{\pm}$  and  $\bar{\Phi}_{\pm} \rightarrow \hat{\Phi}_{\pm}^{\dagger}$ , with the equal time (anti-) commutation relations

$$[\hat{\Phi}_{\pm}(\mathbf{x}), \hat{\Phi}_{\pm}^{\dagger}(\mathbf{y})]_{\pm} = \pm \delta^2(\mathbf{x} - \mathbf{y}). \quad (43)$$

In the physical gauge-invariant subspace of the full Hilbert space, the Hamiltonian is given by

$$\begin{aligned} \hat{H} &= \int \{ -\hat{\Phi}_{+}^{\dagger} (\partial_a - \frac{i}{2} A_a^T)^2 \hat{\Phi}_{+} + \hat{\Phi}_{-}^{\dagger} (\partial_a - \frac{i}{2} A_a^T)^2 \hat{\Phi}_{-} \\ &\quad - 4\pi : (\hat{\Phi}_{+}^{\dagger} \hat{\Phi}_{+})^2 - (\hat{\Phi}_{-}^{\dagger} \hat{\Phi}_{-})^2 : \} d^2x, \end{aligned} \quad (44)$$

where the normal product operator  $: :$  and  $A_a^T = 8\pi \epsilon_{ab} \partial_b^{-1} (\hat{\Phi}_{+}^{\dagger} \hat{\Phi}_{+} - \hat{\Phi}_{-}^{\dagger} \hat{\Phi}_{-})(\mathbf{x})$  have been used. The fields  $\hat{\Phi}_{\pm}$  are local and invariant under local gauge transformations generated by the first-class constraints. Therefore, they may be exploited to study the breaking of global simmetries.

The quantum version of relations (30) leads to  $[\hat{B}(\mathbf{x}), \hat{\eta}(\mathbf{y})] = 8\pi i \delta^2(\mathbf{x} - \mathbf{y})$ , which implies the relation

$$\exp(i \hat{\eta}(\mathbf{y})) \hat{B}(\mathbf{x}) \exp(-i \hat{\eta}(\mathbf{y})) = \hat{B}(\mathbf{x}) + 8\pi \delta^2(\mathbf{x} - \mathbf{y}). \quad (45)$$

This result is used to prove that  $\hat{\Phi}_{\pm}$  are the gauge invariant operators which create a charge-solenoid composite, having magnetic flux equal to  $\pm 8\pi$ .

The Hamiltonian (44) commutes with the particle number operators

$$\hat{N}_{\pm} = \int \hat{\Phi}_{\pm}^{\dagger} \hat{\Phi}_{\pm} d^2x. \quad (46)$$

Hence, we can diagonalize them simultaneously via the Fock representation for a quantum state with  $N = N_+ + N_-$  particles, which is given by

$$|N_+, N_- \rangle = \int \prod_{i=1}^{N_+} d^2 x_i^+ \prod_{j=1}^{N_-} d^2 x_j^- \Psi \left( \mathbf{x}_1^+, \dots, \mathbf{x}_{N_+}^+, \mathbf{x}_1^-, \dots, \mathbf{x}_{N_-}^- \right) \hat{\Phi}_+^\dagger(\mathbf{x}_1^+) \dots \hat{\Phi}_+^\dagger(\mathbf{x}_{N_+}^+) \hat{\Phi}_-^\dagger(\mathbf{x}_1^-) \dots \hat{\Phi}_-^\dagger(\mathbf{x}_{N_-}^-) |\mathbf{0}\rangle, \quad (47)$$

where the vacuum state is defined by  $\hat{\Phi}_\pm |\mathbf{0}\rangle = 0$ . The function  $\Psi$  is an arbitrary element of the Hilbert space  $L_2[\mathcal{R}^{2(N_++N_-)}]$  obeying the Schrödinger equation for  $(N_+ + N_-)$ -bodies. Furthermore, the state (47) is gauge invariant by construction. Thus, we have selected in the full Hilbert space only those states which contain the whole physical information.

Nevertheless, up to now we have dealt with a quite general "anyonic" field theory, without using the constraint  $\gamma$ , which in some sense takes into account the properties of the original spin field model. As a consequence, "quantizing" Eq. (37) in the form

$$\hat{\gamma} = \hat{D}_+ \hat{\Phi}_- (\mathbf{x}) - \hat{D}_- \hat{\Phi}_+ (\mathbf{x}), \quad (48)$$

we should obtain the physical states by requiring that  $\hat{\gamma}|N_+, N_- \rangle = 0$ . However, for arbitrary occupation numbers  $(N_+, N_-)$  this equation leads to complicated relations for the wave function  $\Psi$ . Here we display the result only for the finite boson case  $|N_+, 0\rangle$ , namely

$$\Psi \left( \mathbf{x}_1^+, \dots, \mathbf{x}_{N_+}^+ \right) = \mathcal{F} \left( \bar{z}_1^+, \dots, \bar{z}_{N_+}^+ \right) \prod_{i < j} \left| \mathbf{x}_i^+ - \mathbf{x}_j^+ \right|^{-2}, \quad (49)$$

where  $\mathcal{F}$  is a holomorphic function of its arguments. By using a singular gauge transformation [23-24], the expression (49) takes the form of the Laughlin wave function [25], describing a condensate state of bosonic solitons, which may be related with quantum disorder state of the original ferromagnets [12]. Furthermore, it should be proved that the quantum state corresponding to the wave function (49) is actually an eigenvector of the Hamiltonian (44). This work is in progress, and we have strong indications in the positive sense, because at the classical level the Hamiltonian (23) can be written as a linear combination of the constraint  $\gamma$  and  $\Gamma_1$ .

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## References

- [1] F. A. Zee, *Prog. Theor. Phys. Suppl.* **107**, 77 (1992).
- [2] R. E. Prange and S. M. Girvin, *The Quantum Hall Effect* (Springer-Verlag, New York, 1990).
- [3] S. C. Zhang, T. H. Hansson and S. Kivelson, *Phys. Rev. Lett.* **62**, 82 (1989).
- [4] Z. F. Ezawa, M. Hotta and A. Iwazaki, *Progr. Theor. Phys. Suppl.* **107**, 185 (1992).
- [5] R. Jackiw and S-Y. Pi, *Phys. Rev. Lett.* **64**, 2969 (1990); **66**, 2682 (1990); *Phys. Rev.* **D 42**, 3500 (1990).
- [6] L. Martina, O.K. Pashaev, G. Soliani, *Mod. Phys. Lett. A* **8**, 3241 (1993).
- [7] V. G. Makhankov and O.K. Pashaev, *Integrable Pseudospin Models in Condensed Matter*, in *Sov. Sci. Rev., Math. Phys.* **9**, 1 (1992).
- [8] L. Martina, O.K. Pashaev, G. Soliani, *Phys. Rev.* **B 48**, 15787 (1993).
- [9] T. H. O'Dell, *Ferromagnetodynamics, the dynamics of magnetic bubbles, domains and domain wall*, New York, Wiley (1981).
- [10] N. Papanicolau and T. N. Tomaras, *Nucl. Phys B* **360**, 425 (1991).
- [11] A. A. Belavin and A. M. Polyakov, *JETP Lett.* **22**, 245 (1975).
- [12] A. Zee, *Physics in (2+1)-dimensions* Y.M. Cho ed. (World Scientific, Singapore, 1992).
- [13] M. Greiter, X.G. Wen and F. Wilczek, *Nucl. Phys.* **B374**, 567 (1992), "Paired Hall states in double layer electron system", IASSNS-HEP-92/1
- [14] L. Faddeev, R. Jackiw, *Phys. Rev. Lett.* **60**, 692 (1988).
- [15] P. A. M. Dirac, *Lectures on Quantum Mechanics* (Yeshiva University, New York, 1964).
- [16] P. de Sousa Gerbert, *Phys. Rev.* **D 42**, 543 (1990).
- [17] T. Matsuyama, *Phys. Lett. B* **228**, 99 (1989).
- [18] C. G. Han, *Phys. Rev.* **D 47**, 5521 (1993).
- [19] R. Banerjee, A. Chatterjee and V. V. Sreedhar, *Ann. Phys.* **222**, 254 (1993).
- [20] M. Chaichian and N. F. Nelipa, *Introduction to gauge field theory* (Springer-Verlag, Berlin 1984).
- [21] D. Boyanovsky, E. T. Newman and C. Rovelli, *Phys. Rev.* **D 45**, 1210 (1992); D. Boyanovsky, *Int. J. Mod. Phys. A* **7**, 5917, (1992).
- [22] N. K. Pak and R. Percacci, *J. Math. Phys.* **30**, 2951 (1989); N. K.L Pak, *Progr. Theor. Phys.* **88**, 585 (1992).
- [23] F. Wilczek, *Phys. Rev. Lett.* **48**, 1144 (1982).
- [23] S. Girvin et al., *Phys. Rev. Lett.* **65**, 1489, (1992).
- [25] R.B. Laughlin, *Phys. Rev. Lett.* **50**, 1395 (1983).