Spin quantum Hall effects in a spin-1 topological paramagnet

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AKLT state (or Haldane phase) in a spin-1 chain represents a large class of gapped topological paramagnets, which hosts symmetry-protected gapless excitations on the boundary. In this work we show how to realize this type of featureless spin-1 states on a generic two-dimensional lattice. These states have a gapped spectrum in the bulk but supports gapless edge states protected by spin rotational symmetry along a certain direction, and are featured by spin quantum Hall effect. Using fermion representation of integer-spins we show a concrete example of such spin-1 topological paramagnets on kagome lattice, and suggest a microscopic spin-1 Hamiltonian which may realize it.

I. INTRODUCTION

The fate of frustrated quantum magnets at very low temperatures has for long been an important question in condensed matter physics[1,2]. In contrast to unfrustrated magnets where all spins freeze into an ordered pattern below a certain critical temperature, geometric frustrations and/or quantum fluctuations could strongly suppress the long-range order in frustrated magnets. When these frustrations/fluctuations dominate, magnets refuse to order even down to zero temperature and form a liquid-like cooperative paramagnet. One possibility for such a zero-temperature disordered many-spin state is the quantum spin liquid[3,4], which supports bulk quasiparticles obeying fractional statistics[5–7]. Another possibility is a non-fractionalized featureless spin state, which is gapped in the bulk but hosts exotic gapless excitations on the boundary. A famous example of these "topological paramagnets" is the AKLT state[8,9] in a spin-1 chain[10]. On each end of such an open chain there is a spin-half excitation. Recently it was revealed that the stability of such topological paramagnets (and their boundary excitations) are protected by certain symmetry[11,12]. In other words, in the absence of any symmetry, such a AKLT-type state can be continuously connected to a trivial product state (with no boundary excitations) without closing the bulk energy gap. These non-fractionalized gapped phases are hence coined[13] "symmetry protected topological phases". For instance AKLT state in one spatial dimension can be protected by either $SO(3)$ spin rotational symmetry or time reversal symmetry[12].

IA natural question is: in two spatial dimensions, do similar topological paramagnets exist for integer spins, analogous to one-dimensional AKLT state (or Haldane phase)? The answer is yes. In fact there are an infinite number of different gapped non-fractionalized integer-spin phases, which have gapless edge excitations protected by either $SO(3)$ or $U(1)$ spin rotational symmetry[13–17]. They are also featured by spin quantum Hall effect[14,18], *i.e.* quantized Hall conductance $\sigma_{xy}^s = 2k$, $k \in \mathbb{Z}$ in unit of $\hbar/2\pi$ for spins. To be specific, the spatial gradient of external Zeeman field B_z in one direc-

tion would result in spin current $\vec{j}^{S^z}$ in the perpendicular direction:

$$j_x^{S^z} = -\sigma_{xy}^s \frac{\mathrm{d}B_z(y)}{\mathrm{d}y}. \qquad (1)$$

The gapless edge states and quantized response σ_{xy}^s will be protected as long as $U(1)_{S^z}$ (spin rotation along $\hat{z}$-axis) symmetry is preserved. Notice that *spin quantum Hall effect* here is different from the quantum spin Hall effect[19–21] in time reversal invariant systems, which on the other hand measures the spin response to a transverse electric field. Similar gapped featureless boson states protected by $U(1)$ symmetry (boson charge conservation) have been proposed in the continuum[15,22] and on the lattice[23,24]. However so far a lattice realization of these featureless non-fractionalized states in a integer-spin system is still missing.

In this work we propose a general way to realize these AKLT-like topological paramagnets protected by $U(1)_{S^z}$ symmetry on a two-dimensional lattice. We use the fermion representation[25] of integer spins to construct their many-body wavefunctions. We also derive low-energy effective theory of these featureless gapped phases and their spin quantum Hall responses with $\sigma_{xy}^s = \pm 2$. A careful analysis shows that their edge excitations will remain gapless, robust against any perturbations, as long as $U(1)_{S^z}$ symmetry is not broken. A concrete example of spin-1 magnets on kagome lattice is provided to demonstrate the general construction: the corresponding spin-1 topological paramagnet preserves all kagome lattice symmetries as well as $U(1)_{S^z}$ spin rotational symmetry. Such a state has no net magnetization. Moreover a microscopic Hamiltonians which may realize these states on kagome lattice is proposed.

II. FERMIONIC REPRESENTATION OF $S = 1$ SPINS

For arbitrary spin-S one can introduce $2S + 1$ species of fermions $\{f_m| -S \leq m \leq S\}$ to represent the spin

operator[25,26]:

$$\hat{\mathbf{S}} = f^\dagger \mathbf{I} f, \quad f \equiv (f_S, f_{S-1}, \cdots, f_{-S})^T. \tag{2}$$

where three $(2S+1) \times (2S+1)$ matrices I^a are given by

$$I^a_{m,n} = \langle S, m | \hat{S}^a | S, n \rangle, \quad a = x, y, z.$$

$|S, m\rangle$ stands for the $S^z = m$ eigenstate of a spin-S. In the case of spin-1 $e.g.$ we have

$$\hat{S}^+ \equiv \hat{S}^x + \mathrm{i}\hat{S}^y = \sqrt{2}(f^\dagger_{+1} f_0 + f^\dagger_0 f_{-1}),$$
$$\hat{S}^z = f^\dagger_{+1} f_{+1} - f^\dagger_{-1} f_{-1}. \tag{3}$$

The Hilbert space of these $2S + 1$ species of fermions (2^{2S+1}-dimensional) is larger than the physical $(2S+1)$-dimensional Hilbert space of a spin-S, therefore we need to project the fermion system into the physical Hilbert space of spins. Such a projection is enforced by the following single-occupancy constraint:

$$\hat{N}_f \equiv \sum_{m=-S}^{S} f^\dagger_m f_m = 1. \tag{4}$$

In other words the physical spin-S state $|spin\rangle$ is obtained by projection $\mathbf{P}(\hat{N}_f = 1) \equiv \prod_{\mathbf{r}} \left(1 - \sum_m f^\dagger_{\mathbf{r},m} f_{\mathbf{r},m} \right)$

$$|spin\rangle = \mathbf{P}(\hat{N}_f = 1)|fermion\rangle. \tag{5}$$

on fermion "mean-field" state $|fermion\rangle$. More specifically, the many-body wavefunction $\Phi\big(S^z_{\mathbf{r}} = m_{\mathbf{r}}\big)$ for spin-S operators $\hat{\mathbf{S}}_{\mathbf{r}} = \sum_{m,n} f^\dagger_{\mathbf{r},m} \mathbf{I}_{m,n} f_{\mathbf{r},n}$ on a lattice ($\mathbf{r}$ denotes lattice sites) is given by

$$\Phi\big(\{S^z_{\mathbf{r}} = m_{\mathbf{r}}\}\big) = \langle 0| \prod_{\mathbf{r}} f_{\mathbf{r},m_{\mathbf{r}}} |fermion\rangle. \tag{6}$$

where $|0\rangle$ denotes the vacuum for fermions $\{f_m\}$. The fermion state $|fermion\rangle$ must have the proper filling: $i.e.$ one fermion per site on average, otherwise the projection (5) or (6) will vanish identically.

III. MANY-BODY WAVEFUNCTION AND EFFECTIVE THEORY

Under a spin rotation $\hat{R}^z_\theta \equiv e^{\mathrm{i}\theta\hat{S}^z}$ along $\hat{z}$-axis by an angle θ, the fermions $\{f_m\}$ transform as

$$f_m \to (\hat{R}^z_\theta)^\dagger f_m \hat{R}^z_\theta = e^{\mathrm{i}m\theta} f_m. \tag{7}$$

up to a $U(1)$ gauge redundancy[39] $f_m \to e^{\mathrm{i}\phi} f_m$, $\phi \in [0, 2\pi)$.

We construct the fermion state $|fermion\rangle$ by filling energy levels of the following hopping Hamiltonian of fermions: $H_f = \sum_{m,n=-S}^{+S} \sum_{\mathbf{r},\mathbf{r}'} f^\dagger_{\mathbf{r},m} \mathcal{H}^{m,n}_{\mathbf{r},\mathbf{r}'} f_{\mathbf{r}',n}$. Notice that the number of fermions in the mean-field ground state $|fermion\rangle$, $i.e.$ the number of filled single-particle

energy levels, is equal to the total number of lattice sites, in order to guarantee the constraint (4) on every site. In the presence of $U(1)_{S^z}$ spin rotational symmetry along $\hat{z}$-axis, those hopping terms mixing different species of fermions are forbidden by symmetry (7) and the above hopping Hamiltonian reduces to $\mathcal{H}^{m,n}_{\mathbf{r},\mathbf{r}'} = \delta_{m,n}\mathcal{H}^m_{\mathbf{r},\mathbf{r}'}$

$$H_f = \sum_{m=-S}^{+S} \sum_{\mathbf{r},\mathbf{r}'} f^\dagger_{\mathbf{r},m} \mathcal{H}^m_{\mathbf{r},\mathbf{r}'} f_{\mathbf{r}',m} \tag{8}$$

In order to obtain a gapped spin-S state, we require that each species of fermions f_m to form a band insulator respectively. Constraint (4) implies that on average the filling number of each species of fermions f_m is $\nu = 1/(2S + 1)$ per site.

Now let's focus on a spin-1 system on a lattice $\{\mathbf{r}\}$, which involves 3 species of fermions $\{f_{\mathbf{r},+1}; f_{\mathbf{r},0}; f_{\mathbf{r},-1}\}$ in the fermion representation (3). Since each species of fermions $f_{\mathbf{r},m}$ has a filling $\nu = 1/3$ per site and forms a band insulator, there is a well-defined Chern number[27] C_m (or Hall conductance) for each species of fermions. In the following we derive the effective theory of the spin-S state obtained from projection (6) on the fermion state. For simplicity we first assume the total Chern number for all filled bands of fermion species f_m is $C_m = \pm 1$. Conserved fermion currents J^μ_m can be expressed in terms of dynamical $U(1)$ gauge fields a^m_μ as

$$J^\mu_m = \frac{\epsilon^{\mu\nu\lambda}}{2\pi} \partial_\nu a^m_\lambda, \quad m = -S, -S+1, \cdots, S-1, S.$$

by non-relativistic duality transformation[28,29] in $2 + 1$-dimensions. Summation over repeated indices $\mu, \nu, \lambda = t, x, y$ is always assumed. The fermion band structure is described by the following effective Chern-Simons theory:

$$\mathcal{L}_f = -\frac{\epsilon^{\mu\nu\lambda}}{4\pi} \sum_{m=-S}^{S} C_m \cdot a^m_\mu \partial_\nu a^m_\lambda, \quad C_m = \pm 1.$$

The local constraint (4) can be written in a covariant form:

$$\frac{\epsilon^{\mu\nu\lambda}}{2\pi} \sum_m \partial_\nu a^m_\lambda = \sum_{m=-S}^{S} J^\mu_m = \bar{J}^\mu \equiv \frac{\epsilon^{\mu\nu\lambda}}{2\pi} \partial_\nu \bar{a}_\lambda$$

where $\bar{a}_\mu$ is a non-dynamical (constant) background field, whose field strength $\bar{J}^0 = (\partial_x \bar{a}_y - \partial_y \bar{a}_x)/2\pi$ is a constant $\big($r.h.s. of Eq. (4)$\big)$. Note that due to constraint (4) the fermions f_m are not free fermions: they couple to a dynamical $U(1)$ gauge field. The constraint (4) can be implemented by introducing $U(1)$ gauge field b_μ serving as a Lagrangian multiplier:

$$\mathcal{L}_{constraint} = \frac{\epsilon^{\mu\nu\lambda}}{2\pi} b_\mu \partial_\nu \Big(\sum_{m=-S}^{S} a^m_\lambda - \bar{a}_\lambda \Big).$$

After integrating out the gauge fields b^μ and a^0_μ, we can obtain the low-energy long-wavelength effective theory of an $S = 1$ (spin-1) state (6):

$$\mathcal{L}_{CS} = \mathcal{L}_f + \mathcal{L}_{constraint} = -\frac{\epsilon^{\mu\nu\lambda}}{4\pi} \begin{pmatrix} a^{+1}_\mu \\ a^{-1}_\mu \end{pmatrix}^T \mathbf{K}\, \partial_\nu \begin{pmatrix} a^{+1}_\lambda \\ a^{-1}_\lambda \end{pmatrix}$$

where 2×2 matrix $\mathbf{K}$ is given by

$$\mathbf{K} = \begin{pmatrix} C_{+1} + C_0 & C_0 \\ C_0 & C_{-1} + C_0 \end{pmatrix} \tag{9}$$

In the presence of $U(1)_{S^z}$ spin rotational symmetry, the spin current of S^z component is conserved and we can couple it to an external $U(1)$ "spin gauge field" A^s_μ:

$$\mathcal{L}_{eff} = \mathcal{L}_{CS} - \frac{\epsilon^{\mu\nu\lambda}}{2\pi} A^s_\mu \partial_\nu \Big(\sum_{m=-S}^{S} m \cdot a^m_\lambda \Big). \tag{10}$$

By integrating out the dynamical gauge fields $a^{\pm 1}_\mu$, one can obtains the quantized Hall response for spin (S^z) transport of the $S = 1$ state:

$$\mathcal{L}_{response} = \frac{C_{+1} + C_{-1} + 4C_0}{C_{+1}C_{-1} + C_0(C_{+1} + C_{-1})} \cdot \frac{\epsilon^{\mu\nu\lambda}}{4\pi} A^s_\mu \partial_\nu A^s_\lambda \tag{11}$$

In other words the spin quantum Hall conductance is $\sigma^s_{xy} = \frac{C_{+1} + C_{-1} + 4C_0}{C_{+1}C_{-1} + C_0(C_{+1} + C_{-1})}$ in unit of $\frac{1}{2\pi}$ for this spin-1 state, assuming the Chern numbers take value of $C_m = \pm 1$. As will be discussed later, gapless edge states protected by $U(1)_{S^z}$ symmetry are responsible for the spin quantum Hall effect[14] here.

Notice that for a spin-1 state described by effective theory (10), its ground state degeneracy on a torus[30,31] is $|\det \mathbf{K}|$ in (9). For the case $C_{+1} = C_{-1} = -C_0 = \pm 1$, we have $\det \mathbf{K} = -1$ and hence all the excitations are bosons (*i.e.* no fractionalization) in the corresponding spin-1 state (6). Therefore it is a gapped featureless non-fractionalized spin-1 state with $U(1)_{S^z}$ symmetry. Besides it has gapless edge excitations protected by $U(1)_{S^z}$ symmetry[14,15,22,32], responsible for the spin quantum Hall conductance $\sigma^s_{xy} = \pm 2$.

It's also worth mentioning that when $\mathcal{H}^m_{\mathbf{r},\mathbf{r}'} \equiv \mathcal{H}^0_{\mathbf{r},\mathbf{r}'}$, $\forall m$ in fermion hopping Hamiltonian (8), the projected spin-S state (6) has full $SO(3)$ spin rotational symmetry. When every fermion species has the same Chern number $C_{+1} = C_{-1} = C_0 = \pm 1$, the resultant spin-1 state (6) is a chiral spin liquid[6] with

$$\mathbf{K} = \pm \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}. \tag{12}$$

in its effective theory (10). It has two chiral edge modes and spin quantum Hall effect ($\sigma^s_{xy} = \pm 2$), as well as fractional excitations in the bulk. To be specific there are two different anyon excitations: both have statistical angle $\theta = \frac{2\pi}{3}$, but their mutual braiding statistics is $\theta' = \frac{2\pi}{3}$.

There is a simple physical picture which explains why the choice of Chern numbers $C_{+1} = C_{-1} = -C_0 = \pm 1$ gives rise to a non-fractionalized topological paramagnet. A spin-1 system can always be mapped to a boson system consisting of two species of hard-core bosons γ_+ and γ_-: they carry spin $S^z = \pm 1$ respectively. A $S^z = 0$ spin is mapped to a lattice site with no boson occupancy. The choice of Chern number $C_0 = \pm 1$ simply determines the flux attached to each boson to form composite fermions[33]. These composite fermions are nothing but f_{+1} and f_{-1} in representation (3). A natural question is: imagine we insert an external 2π flux Φ_0 coupled to boson γ_+ (and hence to composite fermion f_{+1}), what's the response of the system? It's straightforward to work out that external flux Φ_0 induces local charge polarizations

$$\delta\rho_{+1}/\Phi_0 = \frac{C_{+1} + C_0}{C_{+1}C_{-1} + C_0(C_{+1} + C_{-1})},$$
$$\delta\rho_{-1}/\Phi_0 = \frac{-C_0}{C_{+1}C_{-1} + C_0(C_{+1} + C_{-1})}.$$

Notice that a fractional charge polarization will imply nontrivial ground state degeneracy[34], hence we need to choose Chern numbers satisfying

$$C_{+1}C_{-1} + C_0(C_{+1} + C_{-1}) = \pm 1.$$

with $C_m = \pm 1$. Therefore one choice to obtain a non-fractionalized topological paragmagnet is $C_{+1} = C_{-1} = -C_0 = \pm 1$. On the other hand, if all fermions share the same Chern number $C_{+1} = C_{-1} = C_0 = \pm 1$, there will be fractional charge polarization $\delta\rho_{-1} = \mp 1/3$ and therefore 3-fold ground state degeneracy on torus.

IV. EDGE EXCITATIONS PROTECTED BY $U(1)_{S^z}$ SYMMETRY

Since the effective Chern-Simons theory (10) for the spin-1 state is obtained, its edge theory is straightforwardly obtained from bulk-edge correspondence[35]. One can easily show[22] that no matter what perturbations are added to the system, there will be gapless edge excitations as long as $U(1)_{S^z}$ symmetry is preserved on the edge. The effective edge theory (the edge is along $\hat{x}$-direction) derived from bulk Chern-Simons theory (10) is

$$\mathcal{L}_{edge} = \frac{1}{4\pi} \sum_{I,J=\pm 1} \left(\mathbf{K}_{I,J} \partial_t \phi_I \partial_x \phi_J - \mathbf{V}_{I,J} \partial_x \phi_I \partial_x \phi_J \right) - \frac{1}{2\pi} \sum_{m\neq 0} m \left(A^s_0 \partial_x \phi_m - A^s_x \partial_t \phi_m \right).$$

where $\mathbf{V}_{I,J}$ is a positive-definite real symmetric matrix which determines the velocity of edge modes. The S^z density for the spin-1 system on the edge is given by

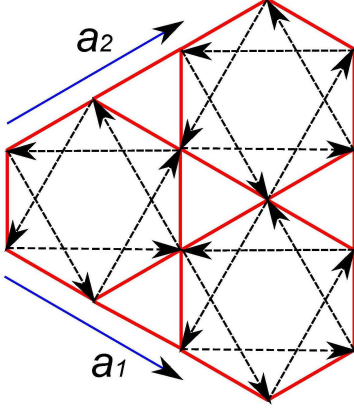$$S^z(x) \simeq \sum_{m=\pm 1} m \frac{\partial_x \phi_m(x)}{2\pi}.$$

FIG. 1: (color online) Hopping Hamiltonian (13) for f_m-fermion of spin-1 magnets on kagome lattice. Solid red lines are 1st nearest neighbor (NN) hopping terms with *real* amplitude t_1^m, while dashed arrows represent 2nd NN hoppings with *imaginary* amplitude $\mathrm{i}t_2^m \cdot C_m \nu_{\mathbf{rr'}}$ between site $\mathbf{r}$ and $\mathbf{r'}$. Here $\nu_{\mathbf{rr'}} = +1$ along on the arrow direction, or -1 along the opposite direction. $C_m = \pm 1$ is the Chern number of the lowest band of this hopping Hamiltonian. $\vec{a}_{1,2}$ denote the two Bravais lattice vectors of kagome lattice and there are three lattice sites in each unit cell.

and this S^z component is in general gapped in the presence of $U(1)_{S^z}$ symmetry. On the other hand, the transverse components of spin-1 on the edge become

$$S^+(x) = S^x(x) + \mathrm{i}S^y(x) \simeq$$
$$e^{\pm \mathrm{i}(2\phi_{+1}+\phi_{-1})} + e^{\mp \mathrm{i}(\phi_{+1}+2\phi_{-1})}.$$

for $\mathbf{K} = \mp \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ in (9). The commutation relation $[S^z(x), S^+(y)] = \delta(x-y)S^+(y)$ is implied by the Kac-Moody algebra $[\phi_{+1}(x), \partial_x \phi_{-1}(y)] = [\phi_{-1}(x), \partial_x \phi_{+1}(y)] = \mp 2\pi \mathrm{i}\delta(x-y)$ of chiral bosons $\{\phi_{\pm 1}\}$ on the edge. The transverse components constitute the gapless excitations on the edge. Under a spin rotation along $\hat{z}$-axis the chiral bosons transform as

$$e^{\mathrm{i}\theta \int S^z(y)\mathrm{d}y}\phi_m(x)e^{-\mathrm{i}\theta \int S^z(y)\mathrm{d}y} = \phi_m(x) \pm m\theta.$$

where $m = \pm 1$ in the spin-1 system. If $U(1)_{S^z}$ symmetry is preserved, the edge excitations $\{\phi_{\pm 1}(x)\}$ must be gapless and transverse spin components $\{S^\pm(x)\}$ have power-law correlations $\langle S^+(x,t)S^-(0,t)\rangle \sim 1/|x|^4$. These edge states will remain gapless unless $U(1)_{S^z}$ symmetry is spontaneously broken and transverse magnetic order is developed on the edge[22]. On the other hand, the correlation functions of S^z component also decay in a power-law fashion but with a different exponent: $\langle S^z(x,t)S^z(0,t)\rangle \sim 1/|x|^2$.

V. EXAMPLES ON KAGOME LATTICE AND MICROSCOPIC HAMILTONIANS

Here we demonstrate the above general construction by an explicit example: $U(1)_{S^z}$-symmetric spin-1 states on kagome lattice with spin quantum Hall effects. Microscopic Hamiltonians which could realize them are also derived from strong-coupling expansion[36,37].

The fermion hoping Hamiltonian (8) for the $U(1)_{S^z}$-symmetric spin-1 states is shown in FIG. 1:

$$H_f = \sum_{m=-1}^{+1} \left(t_1^m \sum_{\langle \mathbf{rr'}\rangle} f_{\mathbf{r},m}^\dagger f_{\mathbf{r'},m} \right. \tag{13}$$

$$\left. -\mu^m \sum_{\mathbf{r}} f_{\mathbf{r},m}^\dagger f_{\mathbf{r},m} + \mathrm{i}t_2^m \cdot C_m \sum_{\langle\langle \mathbf{rr'}\rangle\rangle} \nu_{\mathbf{rr'}} f_{\mathbf{r},m}^\dagger f_{\mathbf{r'},m} \right).$$

where $\langle \mathbf{rr'}\rangle$ and $\langle\langle \mathbf{rr'}\rangle\rangle$ represent 1st and 2nd nearest neighbor pairs. Here $t_1^m, t_2^m < 0$ are real hopping parameters and μ^m is the chemical potential for f_m-fermions. $C_m = \pm 1$ is the Chern number of the lowest f_m-fermion band (see FIG. 2), which changes sign under time reversal. In the fermion state $|fermion\rangle$ in (5)-(6) for spin-1 magnets, we require each species of fermions $\{f_m | m = 0, \pm 1\}$ to fill their lowest band of hopping Hamiltonian (13). This can be done by choosing the chemical potential $\mu^m = t_1^m < 0$ in (13). One can see that on average there are 3 fermions per unit cell, or one fermion per site in $|fermion\rangle$, consistent with constraint (4) for fermion representation (3) of spin-1. When we choose the Chern number of the filled lowest bands to be $C_{+1} = C_{-1} = -C_0 = \pm 1$, after projection (5)-(6) we obtain a non-fractionalized featureless spin-1 state with spin quantum Hall conductance $\sigma_{xy}^s = \pm 2$. Such a featureless state preserves all the lattice symmetries of kagome lattice[40] as well as $U(1)_{S^z}$ spin rotational symmetry along the $\hat{z}$-direction. However it does break time reversal symmetry, by having a quantized spin Hall conductance. It has no magnetization along S^z direction, though.

The above construction on kagome lattice can be generalized to other two-dimensional lattices, as long as filled fermion bands have Chern numbers $C_{+1} = C_{-1} = -C_0 = \pm 1$ and one fermion per site on average. On square lattice *e.g.*, where there is one site per unit cell, in general we need to insert flux in each plaquette[37,38] for fermion hopping Hamiltonian (8) to satisfy the requirement for Chern numbers. Although flux insertion breaks the explicit lattice translation symmetry for fermion hopping Hamiltonian, the spin-1 wavefunction (5)-(6) after projection still preserves lattice symmetries.

The featureless spin-1 state $|spin\rangle$ is obtained from projection (5)-(6) on the fermion state $|fermion\rangle$. This "hard" projection can be implemented in a softer way, by introducing an energy penalty term for violating on-site constraint (4) into fermion Hamiltonian (8). Specifically the following four-fermion interaction term

$$H_{int} = U \sum_{\mathbf{r}} \left(\sum_{m=-1}^{+1} f_{\mathbf{r},m}^\dagger f_{\mathbf{r},m} - 1 \right)^2 = U \sum_{\mathbf{r}} \left(\hat{N}_{\mathbf{r},f} - 1 \right)^2.$$
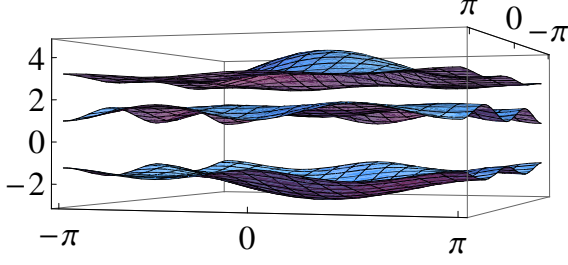
FIG. 2: (color online) Band structure of f_m-fermion hopping Hamiltonian (13) in the 1st Brillouin zone of kagome lattice. The hopping parameters are chosen as $\mu^m = t_1^m = -1$ and $t_2^m = -1/2$. The Chern number for the three bands are $\{C_m, -2C_m, C_m\}$ from bottom to top, with the choice $C_m = \pm 1$.

can enforce the constraint $\hat{N}_{\mathbf{r},f} = 1$ in its low-energy subspace. In the strong-coupling limit $|t_{1,2}^m| \ll U$, the effective Hamiltonian in the zero-energy subspace of H_{int} can be derived from degenerate perturbation theory:

$$H_{eff} = \mathbf{P} H_f \left(1 + \mathbf{Q}\frac{1}{H_{int}}\mathbf{Q}H_f\right)^{-1}\mathbf{P}$$
$$= \mathbf{P}H_f\mathbf{P} - \mathbf{P}H_f\mathbf{Q}\frac{1}{H_{int}}\mathbf{Q}H_f\mathbf{P} + O(\frac{t^3}{U^2})$$

where $\mathbf{P}$ is the projection operator onto the low-energy subspace fulfilling the single-occupancy constraint per site. $\mathbf{Q} = 1 - \mathbf{P}$ projects into the high-energy subspace where constraint (4) is violate. The low-energy subspace satisfying single-occupancy constraint is nothing but the spin-1 Hilbert space and we obtain a spin-1 Hamiltonian which favors the *projected* groundstate of H_f as its groundstate.

For the specific fermion model (13) on kagome lattice, the corresponding effective spin-1 model derived from strong-coupling perturbation theory has the following form (up to order $\sim t^4/U^3$ terms):

$$H_{kagome} = \sum_{\mathbf{r}} \text{Tr}\left[(\Lambda - 4\frac{\mathbf{T}_1\mathbf{T}_1^* + \mathbf{T}_2\mathbf{T}_2^*}{U})\hat{\mathbf{M}}_{\mathbf{r}}\right]$$
$$+\frac{1}{U}\sum_{\langle\mathbf{rr'}\rangle}\text{Tr}\left[\mathbf{T}_1\hat{\mathbf{M}}_{\mathbf{r}}\mathbf{T}_1^*\hat{\mathbf{M}}_{\mathbf{r'}}\right] + \frac{1}{U}\sum_{\langle\langle\mathbf{r,r'}\rangle\rangle}\text{Tr}\left[\mathbf{T}_2\hat{\mathbf{M}}_{\mathbf{r}}\mathbf{T}_2^*\hat{\mathbf{M}}_{\mathbf{r'}}\right]$$
$$+\frac{1}{U^2}\left(\sum_{\triangleright}\mathcal{R}_{\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3} + \sum_{\triangledown}\mathcal{R}_{\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3} + \sum_{\triangle}\mathcal{R}_{\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3}\right)$$
(14)

where $\Lambda = \text{diag}(\mu^m) \equiv \text{diag}(\mu^{+1}, \mu^0, \mu^{-1})$ is the chemical potential matrix, while $\mathbf{T}_1 = \text{diag}(t_1^m)$ and $\mathbf{T}_2 = \text{diag}(iC_m t_2^m)$ are 1st and 2nd NN fermion hopping matrices. Matrix $\hat{\mathbf{M}}_{\mathbf{r}}$ is defined in terms of local spin operators (or on-site fermion bilinears):

$$\left(\hat{\mathbf{M}}_{\mathbf{r}}\right)_{l,m} \equiv f_{\mathbf{r},l}f_{\mathbf{r},m}^\dagger = \begin{pmatrix} \frac{2-S_{\mathbf{r}}^z(S_{\mathbf{r}}^z+1)}{2} & \frac{-S_{\mathbf{r}}^- S_{\mathbf{r}}^z}{\sqrt{2}} & \frac{-(S_{\mathbf{r}}^-)^2}{2} \\ \frac{-S_{\mathbf{r}}^z S_{\mathbf{r}}^+}{\sqrt{2}} & (S_{\mathbf{r}}^z)^2 & \frac{S_{\mathbf{r}}^z S_{\mathbf{r}}^-}{\sqrt{2}} \\ \frac{-(S_{\mathbf{r}}^+)^2}{2} & \frac{S_{\mathbf{r}}^+ S_{\mathbf{r}}^z}{\sqrt{2}} & \frac{2-S_{\mathbf{r}}^z(S_{\mathbf{r}}^z-1)}{2} \end{pmatrix}.$$

The 1st line of this Hamiltonian represents on-site S^z anisotropy terms, compatible with the $U(1)_{S^z}$ spin rotational symmetry. The 2nd line are two- and four-spin interactions between 1st and 2nd NNs. They reduce to the $SU(3)$ bilinear-biquadratic form $\vec{S}_{\mathbf{r}}\cdot\vec{S}_{\mathbf{r'}} + (\vec{S}_{\mathbf{r}}\cdot\vec{S}_{\mathbf{r'}})^2$, when three species of fermions $\{f_0, f_{\pm 1}\}$ have exactly the same hopping parameters in (13). The 3rd line corresponds to ring exchange terms in three different types of 1st/2nd NN triangles:

$$\mathcal{R}_{\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3} = \text{Tr}\left[\hat{\mathbf{M}}_{\mathbf{r}_2}\mathbf{T}_{\mathbf{r}_2\mathbf{r}_3}\hat{\mathbf{M}}_{\mathbf{r}_3}\mathbf{T}_{\mathbf{r}_3\mathbf{r}_1}(\mathbf{E}-\hat{\mathbf{M}}_{\mathbf{r}_1})\mathbf{T}_{\mathbf{r}_1\mathbf{r}_2}\right].$$

where $\mathbf{E}$ represents a 3×3 identity matrix. $\mathbf{T}_{\mathbf{rr'}}$ denotes the 3×3 fermion hopping matrix from site $\mathbf{r}$ to $\mathbf{r'}$. Under time reversal operation, the spin-1 operators transform as

$$\hat{\mathbf{M}}_{\mathbf{r}} \to \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\hat{\mathbf{M}}_{\mathbf{r}}\begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

It's straightforward to show that imaginary 2nd NN hoppings with Chern number $C_{+1} = C_{-1} = -C_0 = \pm 1$ and $t_2^m < 0$ will lead to no time reversal symmetry in the spin-1 Hamiltonian H_{kagome}. Both the 2nd and 3rd line in model (14) break the time reversal symmetry. On the other hand, this spin-1 Hamiltonian does have $U(1)_{S^z}$ spin rotational symmetry, and all the kagome lattice symmetries.

From strong-coupling perturbation theory, we only derive the spin-1 Hamiltonian H_{kagome} up to $O(t^4/U^3)$ terms in t/U expansion. A careful study of spin-1 Hamiltonian (14) and the effects of higher-order terms in perturbation expansion are beyond the scope of this paper and we leave them for future studies.

VI. CONCLUSIONS

In this work we show that a class of gapped featureless non-fractionalized spin-1 states, *i.e.* spin-1 topological paramagnets, in analogy to AKLT state in a spin-1 chain, can be constructed on a generic two-dimensional lattice. Using the fermion representation of integer-spins we can write down the many-body wavefunction of these states, and derive microscopic Hamiltonians which may realize them on the kagome lattice. The bulk effective theory of these gapped phases are obtained, along with gapless edge excitations protected by $U(1)_{S^z}$ symmetry and spin quantum Hall conductance $\sigma_{xy} = \pm 2$ in unit of $\hbar/2\pi$. A concrete example is shown for spin-1 magnets on the two-dimensional kagome lattice, which preserves all the kagome lattice symmetries and has no magnetization.

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[39] Notice that all spin operators in (2) are invariant under this $U(1)$ gauge transformation: $f_m \to e^{i\phi} f_m$, $\phi \in [0, 2\pi)$ for $\forall\, m$.

[40] Here the usual mirror reflection symmetry is defined in a slightly different way: a mirror reflection operation along *e.g.* the horizontal line in FIG. 1 is followed by an anti-unitary time reversal operation.