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Anyonization of Lattice Chern–Simons Theory

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We formulate Hamiltonian lattice Chern–Simons theory which has the property that the Chern–Simons gauge fields of the theory can be eliminated by making matter fields multi-valued operators with anyonic statistics. We prove that, when the statistics parameter is an odd integer so that the anyons are bosons, the ground state, which consists of a condensate of bound pairs of flux tubes and fermions, breaks phase invariance. The ensuing long-range order implies that the system is an unconventional superfluid. We formulate a condition which may be useful as a numerical signal for symmetry breaking in the ground state for any statistics parameter. We also discuss an exotic lattice Chern–Simons theory, which makes explicit the relation of anyons to *framed* knot invariants. We discuss various lattice representations of the Chern–Simons term and find the unique local lattice Chern–Simons term with the appropriate naive continuum limit, which permits anyonization.

INTRODUCTION

Anyons are particles that may have "any" spin and statistics, rather than the usual set of integer and half-integer spins that arise in most quantum systems [1–20]. They arise only in $2+1$ dimensions, where the rotation group is abelian and therefore permits representations where the angular momentum is not integer or half-integer [15–20].

Such particles are interesting because they arise in real systems. The anyon hypothesis correctly predicts the form of the Hall coefficient as function of magnetic field in the fractional quantum Hall effect [21–22] and may appear in other quasi-two-dimensional condensed matter systems, such as high T_c superconductors [23–29]. They have recently been the focus of much study.

Anyons may be realized in a theory containing ordinary (charged) fermions or bosons by attaching to these particles a tube of magnetic flux [2, 30]. Since the resulting composites carry both an electric charge and magnetic flux, when two such bound states encircle each other their wave functions acquire an Aharonov–Bohm phase. If one exchanges two particles by adiabatic transport, this phase appears in addition to the ordinary factor ± 1 from bosonic or fermionic statistics.

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It may be adjusted in strength by adjusting the field inside the flux tube and may even be so strong as to change a fermionic particle exchange into a bosonic one (or vice versa). In general, then, the composite particles have effective statistics which can be in between fermi and bose and are called anyons.

The pinning of this magnetic flux tube to a particle may be achieved in the context of quantum field theory as well [8, 9, 11–14, 31, 35, 36]. This is done by coupling matter fields to an abelian gauge field, which itself has no kinetic term of the usual type, but instead is self-coupled through a Chern–Simons term, with the Lagrangian

$$\mathscr{L} = \frac{1}{4\pi\alpha} \varepsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda - A_\mu j^\mu. \tag{0.1}$$

The coupling constant α determines the statistics of the anyons and is usually called the statistics parameter. With this coupling the gauge constraint is

$$2\pi\alpha j^0 - B = 0, \tag{0.2}$$

precisely pinning flux $2\pi\alpha$ to a unit of matter charge. There are no dynamics for this gauge field, since the gauge field appears in the Hamiltonian only through its coupling to matter, and it may therefore be completely eliminated by solving the accompanying constraints [36]. Chern–Simons gauge theory is thus an important tool in the study of anyons. Additionally, we note that if we had added the conventional Maxwell term to (0.1) a photon having a topological mass [15, 37] would appear. In the limit as the coefficient of the Maxwell term vanishes, the mass of the physical photon goes to infinity, leaving the kinematical Chern–Simons interaction.

A quantized Chern–Simons theory coupled to classical charged particles is exactly solvable [38]. In planar (or spherical) geometry it has no physical degrees of freedom and the wavefunction is a pure phase expressing the braiding of the classical particle histories [11]. In more complicated geometries its Hilbert space is finite dimensional and its wavefunctions are related to the conformal blocks of rational conformal field theory [39].

Also, for nonrelativistic quantum mechanical matter with a finite number of particles [40–42] the mapping of fermions to anyons is now well understood and can even be done at the operator level. It is known how to eliminate the gauge interactions in favor of multi-valued wave functions. In operator language, it is possible to construct multi-valued operators which create particles in multi-valued quantum states. The ideas involved have important applications to the study of the quantum Hall effect.

However, in a relativistic quantum field theory, where the matter degrees of freedom are necessarily described by quantum fields, the exact correspondence between a Chern–Simons theory and anyons is less well established. The semiclassical reasoning whereby the Gauss' law constraint (0.2) associates magnetic flux with electric charge is obscured by the fact that in a relativistic field theory it is generally

not possible to diagonalize the charge density operator.[1] Therefore the charge density operator has large fluctuations, as does the magnetic field operator if the constraint (0.2) is obeyed, and the semiclassical arguments relating statistics to abiabatic Aharonov–Bohm phases are not obviously correct. On the other hand, rigorous mathematical arguments have been given for the existence of anyonic particles in relativistic quantum field theories [13, 14]. In order to make these arguments more useful and to better understand their domain of applicability, much work remains to be done.

For example, it would be desirable to complete the analogy between the Chern–Simons theory–anyon correspondence and the familiar fermion–boson mapping in two-dimensional quantum field theory. Especially at the operator level, it would be interesting to construct the analog of a vertex operator, i.e., an "anyon operator" which creates multi-valued anyon states. One could pursue this analogy by constructing an effective Hamiltonian for the anyon operators and thus implementing "anyonization" of the field theory, analogous to bosonization in two dimensions.

This possibility was originally examined for field theories in the continuum [8, 9, 12, 17, 20] and later on the lattice [31–35, 43, 44]. The continuum work for the most part ignored difficulties with regularization which are known to be crucial to the Coleman–Mandelstam [45–46] bosonization of two-dimensional field theories. Also, the precise definition of the multi-valued operators involved, particularly the "singular everywhere" gauge transformations introduced in [8] (and discussed in [47]), is obscure in the continuum.

The latter may be given a precise definition, and all the ultraviolet singularities may be regulated, by putting the field theory on a lattice. In Refs. [43, 44] the construction of anyons from a lattice regulated Chern–Simons theory (originally suggested for the continuum in [8]) was carried out rigorously. It was shown that in a particular lattice fermion–Chern–Simons theory, multi-valued anyon fields could be constructed and the gauge interactions eliminated from their effective Hamiltonian. It should be emphasized that, as in the case of bosonization in two dimensions, this "anyonization" does not necessarily solve the field theory.[2] It simply presents the regulated bare field theory in terms of operators with different statistics. Generally, three-dimensional field theories are much more difficult to solve than two-dimensional ones. This higher complexity of structure, however, is rewarded by the availability of a greater variety of physical phenomena.

In the case of the fermion–Chern–Simons theory with statistics parameter $\alpha =$ odd integer, the Hamiltonian is written in terms of bosonic operators, and the gauge interactions decouple completely [43]. The resulting bosonic operators obey

[1] The transformation between the basis where the Hamiltonian is diagonal and the charge density operator is diagonal is not unitarily implementable even in a free field theory.

[2] To our knowledge, this transformation has not yet been successfully used to obtain an exact solution of any nontrivial interacting three-dimensional field theory.

a spin algebra. This fact was used to map the original Chern–Simons–fermion theory onto the quantum X–Y model (see also [48, 31, 35]). Previous results [49, 50] that the ground state of the two-dimensional quantum X–Y model has long-range order were used in [43] to prove that, in this particular case, the fermion–Chern–Simons theory has a superfluid ground state. The latter fact is most interesting in that the condensate is unconventional, consisting of fermion–vortex pairs rather than pairs of elementary matter fields.

When anyon statistics are neither Bose nor Fermi, the Hamiltonian for the anyon operators contains a residual interaction which can not be eliminated. In Ref. [51] it was shown that anyonic theories of necessity have a local symmetry among its fields which is a representation of the braid group, which breeds a kinematic phase in the Hamiltonian as an interaction—a direct derivation of this result from Chern–Simons theory was found. For rational statistics parameters it consisted of a Z_N gauge interaction.

In the present paper we elaborate and extend that work. We discuss alternative versions of the lattice Chern–Simons term. We present arguments that the Chern–Simons term given here is the *unique* local lattice expression which describes the geometry of a line bundle over a rectangular lattice. We present a detailed description of the solution of the gauge constraints in a Chern–Simons theory and apply them to our Hamiltonian lattice model. We give a construction of the multi-valued angle function which is later used to form multi-valued operators. We also discuss the role of gauge invariance and show that gauge invariance itself is almost sufficient to determine the monodromy of anyon wavefunctions.

In addition, we discuss the possibility of proving that anyons have a superfluid ground state. We give a numerical test for superfluidity and superconductivity there.

Finally, the machinery developed herein will allow us to treat another, more exotic implementation of lattice Chern–Simons theory first introduced by Kantor and Susskind [54], whose advantage is that it simplifies considerably the formalism that we have presented.

This paper is organized as follows. In Section I we introduce our lattice notation. Section II is devoted to representing the Chern–Simons term on the lattice. Much previous work has been done on this and related subjects—in this section we also discuss the relationship between our formulation and these others [52, 13, 32, 33, 31]. We then turn to the problem of constructing anyonic operators. This construction requires the use of the angle function, so Section III is a small detour which delineates how this is done on a lattice. In Sections IV and V, the lattice anyon operators are actually constructed, in two ways. The standard way consists of multiplying the fermion operator with both a vortex creation operator and a multi-valued statistics inducing operator, and it is done in the continuum because of its rotation properties. Afterwards, we show that this may be more neatly achieved by simply attaching a Wilson line to the fermion operator (although this does what we want only for a specific choice of the parameter freedom that we found in the earlier sections). At this time, we also point out the relations to knot theory and the

geometric formulation of this lattice Chern–Simons theory. Section VI discusses the relation of lattice Chern–Simons theory to spin models and its implications for the question of long-range order for anyons. Finally, Section VII uses the methods of Section V to treat the Kantor–Susskind version of lattice Chern–Simons theory.

I. Notation

We work in the Hamiltonian formalism, where time is continuous and where space is a finite two-dimensional square lattice with lattice spacing 1. We define the forward shift operator S_i, $i=1, 2$, by

$$S_i f(x) = f(x+\hat{\imath}) \tag{1.1}$$

and its inverse is denoted by

$$S_i^{-1}: S_i^{-1} f(x) = f(x-\hat{\imath}). \tag{1.2}$$

(For convenience of notation, also define $S_0 = 1$, so that we may use the Greek index notation S_μ, $\mu = 0, 1, 2$). On lattices, differentiation becomes differencing, and it may do so in one of two possible ways—either with forward differencing,

$$d_i f(x) = f(x+\hat{\imath}) - f(x), \qquad d_i = S_i - 1, \tag{1.3}$$

or backward differencing, denoted by

$$\hat{d}_i f(x) = f(x) - f(x-\hat{\imath}), \qquad \hat{d}_i = 1 - S_i^{-1} = S_i^{-1} d_i. \tag{1.4}$$

(Again, for convenience, define $\hat{d}_0 = d_0 = \partial_0$. We will also use the notation $\dot{f} = \partial_0 f$.) Note that summation by parts on a lattice takes the form (neglecting surface terms)

$$\sum_x f(x)\, d_\mu\, g(x) = -\sum_x \hat{d}_\mu f(x)\, g(x), \tag{1.5}$$

by virtue of the lattice Leibniz rule,

$$d_\mu(fg) = f d_\mu g + d_\mu f S_\mu g \tag{1.6}$$

(no sum on μ). The spatial components $A_i(x)$ of the gauge field are real-valued functions on the links specified by the pair $[x, \hat{\imath}]$, and the time component $A_0(x)$ is a function on sites. Thus the field strength tensor in this formalism has two parts. The space–space part,

$$F_{ij}(x) = d_i A_j(x) - d_j A_i(x), \tag{1.7}$$

is a function on plaquettes, but mapped by our use of forward differencing to a function on sites as well, by the convention of associating a plaquette to the site at its lower left corner. The space–time components

$$F_{0i}(x) = d_0 A_i(x) - d_i A_0(x) \tag{1.8}$$

are a function on links. Defining the dual field strength in the usual way,

$$\tilde{F}_\mu(x) = \varepsilon_{\mu\nu\lambda} F_{\nu\lambda}(x), \tag{1.9}$$

the Bianchi identity takes the form

$$d_\mu \tilde{F}_\mu = 0 \tag{1.10}$$

(i.e., using forward differencing).

The lattice fourier transform is defined by

$$f(x) = \int_{\Omega_B} \frac{d^2k}{(2\pi)^2} e^{-ik\cdot x} f(k), \tag{1.11}$$

where

$$\Omega_B = \{(k_1, k_2): -\pi < k_1, k_2 \leqslant \pi\} \tag{1.12}$$

is the Brillouin zone. Furthermore, the fourier transform of the difference and shift operators are

$$d_i(k) = S_i(k) - 1 = e^{-ik_i} - 1, \qquad \hat{d}_i(k) = 1 - S_i^{-1}(k) = 1 - e^{ik_i} \tag{1.13}$$

and obey the identities $-\hat{d}_i(k) = d_i(-k) = d_i^*(k)$, $S_i^{-1}(k) = S_i(-k) = S_i^*(k)$. It is straightforward to derive the equations (with no sum on i)

$$d_i \hat{d}_i = d_i - \hat{d}_i \tag{1.14}$$

and

$$d^2 \hat{d}^2 = (d \times \hat{d})^2 + (d \cdot \hat{d})^2 \tag{1.15}$$

which are useful in deriving some of the formulae in Section III.

Similarly, the time domain has the usual fourier transform

$$f(t) = \int_{-\infty}^{\infty} \frac{dk_0}{2\pi} e^{-ik_0 t} f(k_0) \tag{1.16}$$

and derivatives are defined analogously.

II. Lattice Representations of the Chern–Simons Term

We shall consider a system of fermions coupled to a gauge field, where the only gauge field kinetic term in the action is a Chern–Simons term. We write $L = L_m + L_{CS}$, with the matter Lagrangian minimally coupled to the gauge field,

$$\begin{aligned} L_m = &\sum_x \Psi^\dagger(x, t)(i\partial_t + A_0(x, t))\, \Psi(x, t) \\ &- \sum_x A_0 \rho_0 + \sum_{x,i} \Delta_i(x)\, \Psi^\dagger(x + i, t)\, e^{iA_i(x,t)} \Psi(x, t) + \text{h.c.} \end{aligned} \tag{2.1}$$

ρ_0 is a constant background charge density. $\Delta_i(x)$ is a hopping amplitude which may be chosen arbitrarily and does not affect our calculations, but we note $\Delta_i(x)=1$ corresponds to nonrelativistic fermions, and $\Delta_i(x)=\text{const}(-1)^{\sum_1^i x_j}$ to relativistic fermions, so that either may be treated in this formalism. ($\Delta_i(x)$ could be considered a background gauge field.) The gauge transformation is

$$\Psi(x, t) \to \exp(i\Lambda(x, t))\, \Psi(x, t) \tag{2.2}$$

$$A_\mu(x, t) \to A_\mu(x, t) + d_\mu \Lambda(x, t). \tag{2.3}$$

The Chern–Simons Lagrangian L_{CS} requires more thought. We begin by briefly reviewing previous formulations. Chern–Simons theory on a lattice has been receiving attention for some time now. Originally, Fröhlich and Marchetti [13] used a formalism for differential forms on a lattice (previously developed by Becher and Joos [52]) to write down a simple, local lattice Chern–Simons term and used it to study the quantization of vortices in three-dimensional gauge theory. Lüscher [32] used the same representation of the lattice Chern–Simons term to carry out a careful fermion–boson mapping in the context of Maxwell–Chern–Simons theory (i.e., including the kinetic term $\frac{1}{4}F^2$ for the photon). Later, Müller [33] extended that work to anyons and discussed the role of the braid group and braid statistics.

These theories are interesting in their own right, but our interest is in pure Chern–Simons theory. And while one might expect, from the discussion in the Introduction, that the $e \to \infty$ limit of these theories, in which the photon decouples from the theory, would now be a satisfactorily regulated pure Chern–Simons theory, it is not. This lattice formulation of Chern–Simons theory introduces new singularities which appear only as the Maxwell term disappears (and thus do not cause a problem in Refs. [13, 32, 33]). It is instructive for our purposes to see why.

Their Chern–Simons Lagrangian was, in terms of the dual field strength $\tilde{F}_\mu = \varepsilon_{\mu\nu\lambda} F_{\nu\lambda}$,

$$\mathscr{L}_{CS} = \frac{1}{2\pi\alpha} \sum_x A_\mu(x)\, S_\mu \tilde{F}_\mu(x) \tag{2.4}$$

which, by virtue of the lattice Leibniz rule (1.5) is gauge invariant, on summation by parts and use of the Bianchi identity $d_\mu \tilde{F}_\mu(x) \equiv 0$. Writing (2.4) in k-space yields

$$\mathscr{L}_{CS} = \frac{1}{2\pi\alpha} \left(\sum_{k,i,j} \dot{A}_i(k)\, K_{ij}(k)\, A_j(-k) + \sum_k A_0(-k)\, J(-k)\, B(k) \right) \tag{2.5}$$

with

$$J = -\tfrac{1}{2}(1 + S_1^{-1} S_2^{-1}) \tag{2.6}$$

and

$$K_{ij} = \frac{1}{2} \begin{pmatrix} 0 & S_1^{-1} + S_2 \\ -(S_2^{-1} + S_1) & 0 \end{pmatrix} \tag{2.7}$$

(see footnote before Eq. (2.16)). We see then that both K_{ij}, which determines the commutator function,

$$i\pi\alpha K_{ij}^{-1}(k) = [A_i(k), A_j(-k)], \tag{2.8}$$

and J, which determines the Gauss law,

$$\mathcal{G}(k) = 2\pi\alpha\rho(k) + J(-k)\, B(k) = 0, \tag{2.9}$$

have the same zeroes, at $J = -\frac{1}{2}(1 + \exp - i(k_1 + k_2)) = 0$, i.e., at $k_1 + k_2 = \pm\pi$. The Gauss law, then, forces the vanishing of the charge density at these lines, the "antiferromagnetic modes" in the Brillouin zone. Furthermore, the fourier transform of $K_{ij}^{-1}(k)$ diverges, indicating that K is not invertible as is and there is another constraint which must be dealt with before we can find the commutator in (2.8). The presence of this constraint, which is not a part of any a priori theory of anyon kinematics, indicates that the photon does not fully decouple. Instead it leaves behind a remnant strong interaction, ruins the connection between pure Chern–Simons theory on a lattice and lattice anyons, and prevents their study in this setting.

A further symptom of the difficulties with this form of pure Chern–Simons theory on a lattice may be found in the gauge field propagator. Writing

$$\mathcal{L}_{\mathrm{CS}} = A_\mu(k)\, G_{\mu\nu}^{-1}(k)\, A_\nu(-k) \tag{2.10}$$

with the covariance

$$G_{\mu\nu}^{-1}(k) = \frac{1}{2\pi\alpha}\, \varepsilon_{\mu\lambda\nu}\, d_\lambda(k)\, S_\mu(k) \tag{2.11}$$

(no sum on μ), we see that the classical current–current correlator,

$$\langle j_\mu(k)\, j_\mu(-k)\rangle = G_{\mu\nu}(k) \tag{2.12}$$

is proportional to the inverse of the eigenvalues of G^{-1}, which are (ignoring the inevitable zero eigenvalue from gauge invariance, which can be dealt with by conventional gauge fixing)

$$\pm\frac{1}{2\pi\alpha}\sqrt{d_0^2 + 2 - \sum_i \cos k_i}\,\left|1 + \exp i\sum_i k_i\right|. \tag{2.13}$$

These vanish at $k_1 + k_2 = \pm\pi$. These line segments in the Brillouin zone have co-dimension 1 and therefore bring logarithmic singularities into the fourier transform of $G_{\mu\nu}(k)$, even after the usual gauge degeneracy is dealt with. Thus, the covariance in the Frölich–Marchetti Chern–Simons term is not invertible and cannot be used for pure Chern–Simons theory (without imposing additional constraints, beyond those of gauge invariance).

Contemporaneous with the work of Lüscher was a treatment by Fradkin

[31, 35] of pure lattice Chern–Simons theory. Though Fradkin's constraint is non-singular, his implementation of the Chern–Simons term on the lattice, different from that of Fröhlich–Marchetti, was not gauge invariant. The expression that appeared in that work

$$\mathscr{L}_{\mathrm{CS}} = A_\mu(x)\,\tilde{F}_\mu(x), \qquad \tilde{F}_\mu = \varepsilon_{\mu\nu\lambda}\,d_\nu A_\lambda, \tag{2.14}$$

under a gauge transformation becomes $d_\mu \Lambda(x)\,\tilde{F}_\mu(x)$, which does not realize the Bianchi identity expression $\Lambda(x)\,d_\mu \tilde{F}_\mu(x) \equiv 0$ on summation by parts, but instead $\Lambda(x)\,\hat{d}_\mu \tilde{F}_\mu(x) \neq 0$. Thus a canonical analysis of Fradkin's Chern–Simons term would require second-class constraints and has thus far not been completed in detail.

It is possible to find a suitable modification of the Fröhlich–Marchetti Chern–Simons term which does not have this difficulty and, as we shall see later, still has the property of locality. We begin by seeking a quadratic function of the gauge fields that is odd under P and T, even under PT, at most linear in time derivatives and in A_0, with this latter being the Lagrange multiplier for a Gauss law appropriate to Chern–Simons theory (as in Eq. (2.20) below). We allow for nonlocality on the lattice. Thus,

$$\mathscr{L}_{\mathrm{CS}} = \frac{1}{2\pi\alpha} \sum_{x,y,i} A_0(x,t)\,J_i(x-y)\,A_i(y,t) + \frac{1}{2\pi\alpha} \sum_{x,y,i,j} \dot{A}_i(x,t)\,K_{ij}(x-y)\,A_j(y,t). \tag{2.15}$$

Without loss of generality, we take $K_{ij}(x-y) = -K_{ji}(y-x)$,[3] or

$$K_{ij}(k) = -K_{ji}^*(k). \tag{2.16}$$

Again, the gauge generator and commutator in Hamiltonian quantization are directly determined by the functions J_i and K_{ij}, respectively, as

$$\mathscr{G}(x) = 2\pi\alpha(\rho(x) - \rho_0) + \sum_{y,i} J_i(x-y)\,A_i(y) \tag{2.17}$$

$$[A_i(x), A_j(y)] = i\pi\alpha K_{ij}^{-1}(x-y). \tag{2.18}$$

Gauge invariance requires

$$d_i(k)\,J_i(k) = 0 \tag{2.19a}$$

$$2\hat{d}_i(k)\,K_{ij}(k) = J_j(k). \tag{2.19b}$$

The first equation may be solved by $J_i(k) = d_i^\perp J(k)$ (Given any spatial vector V_i, we define $V_i^\perp = \varepsilon_{ij}\,V_j$), and in terms of this new function $J(k)$, the gauge constraint in k-space is

$$\mathscr{G}(k) = 2\pi\alpha(\rho(k) - \rho_0) - J(-k)\,B(k). \tag{2.20}$$

[3] This condition ensures the property of reflection positivity which is emphasized as a necessary condition in Ref. [15].

We seek a solution to Eq. (2.20) which is nondegenerate, i.e., for which $J(k) \neq 0$ for all k. Equation (2.20b) may be solved by decomposing $K_{ij}(k)$ as

$$K_{ij}(k) = K_{\parallel\parallel}\hat{d}_i d_j + K_{\perp\perp}\hat{d}_i^{\perp} d_j^{\perp} + K_{\perp\parallel}\hat{d}_i^{\perp} d_j + K_{\parallel\perp}\hat{d}_i d_j^{\perp} \tag{2.21}$$

with

$$K_{\parallel\perp}^{*} = -K_{\perp\parallel}, \qquad K_{\parallel\parallel} = -K_{\parallel\parallel}^{*}, \qquad K_{\perp\perp} = -K_{\perp\perp}^{*}, \tag{2.22}$$

from (2.16). Then (2.19b) implies

$$K_{\parallel\parallel} = 0, \qquad K_{\perp\parallel} = -\frac{J^{*}}{2d\cdot d}, \qquad K_{\parallel\perp} = \frac{J^{*}}{2\hat{d}\cdot\hat{d}}; \tag{2.23}$$

$K_{\perp\perp}$ is arbitrary. The determinant of the K which solves (2.19b) is $\det K = \frac{1}{4} J^{*} J$, which implies that the canonical structure has the same degeneracies as the gauge generator, and a nondegenerate choice of $J(k)$ eliminates the degeneracies in both.

In what follows we find it necessary to specialize to the case $J(k) = 1$, leading to the Gauss law,

$$\mathcal{G}(k) = 2\pi\alpha(\rho(k) - \rho_0) - B(k), \tag{2.24}$$

because anyonization requires a local Gauss law.

What remains is the freedom to fix the function $K_{\perp\perp}$. We see that this amounts to the freedom to add a nonlocal but manifestly gauge invariant term of the form

$$\sum_{x,y} K_{\perp\perp}(x-y)\, B(x)\, \dot{B}(y). \tag{2.25}$$

We determine $K_{\perp\perp}$ such that exact anyonization is possible.

It proves convenient to use a part of $K_{\perp\perp}$ to subtract off the trace of K_{ij}—this is useful both in taking the continuum limit and in comparing our implementation of the Chern–Simons term on the lattice with that of Fröhlich and Marchetti [13]. Furthermore, this trace subtraction comes automatically from the requirement of exact anyonization, so that including it now serves only to simplify calculations later. Thus we write

$$K_{\perp\perp} \to K_{\perp\perp} - \frac{\operatorname{tr} K_{ij}}{d\cdot\hat{d}}, \qquad \frac{\operatorname{tr} K_{ij}}{d\cdot\hat{d}} = \frac{d\times\hat{d}}{d\cdot\hat{d}}\left(\frac{1}{d^2} + \frac{1}{\hat{d}^2}\right) \tag{2.26}$$

and, after some rearranging, we may now write K_{ij} as

$$K_{ij} = \frac{d_i d_j^{\perp} - \hat{d}_i^{\perp}\hat{d}_j}{2d\cdot\hat{d}} + K_{\perp\perp}\hat{d}_i^{\perp} d_j^{\perp} \tag{2.27}$$

and K_{ij}^{-1} as

$$K_{ij}^{-1} = \frac{2}{d\cdot\hat{d}}(\hat{d}_i^{\perp}\hat{d}_j - d_i d_j^{\perp}) + 4K_{\perp\perp} d_i \hat{d}_j. \tag{2.28}$$

Note that this trace subtracting piece satisfies the requirement that it be pure imaginary in k-space. Also note that this gives us the relation $\mathrm{Tr}\, K = d \cdot \hat{d} K_{\perp\perp}$.

Finally, we note here that is is possible to show that the naive continuum limit of this lattice representation of the Chern–Simons term has a local continuum limit. To do so, we may write the continuum Chern–Simons term as

$$\mathscr{L}_{\mathrm{CS}} = \varepsilon_{\mu\nu\lambda} A_\mu(x)\, \partial_\nu A_\lambda(x) = A_0(x)\, B(x) + \varepsilon_{ij} A_i(x)\, \dot{A}_j(x) \tag{2.29}$$

and introduce the identity

$$\varepsilon_{ij}\delta(x-y) = \frac{\partial_i \partial_j^\perp - \partial_i^\perp \partial_j}{\partial_i^2}(x-y) \tag{2.30}$$

into the second term; note the similarity with the form equation (2.27). We then see, for Fourier-transformable field configurations at least, that this lattice representation of the Chern–Simons term has a close correspondence with that in the continuum.

III. Definition of the Lattice Angle Function

Because anyons acquire a phase upon circling each other and returning to their original configuration, their wavefunctions are of necessity multi-valued functions of position, and the operators that create them are similarly multi-valued. In fact, the phases accumulated thereby must be an abelian representation of the fundamental group of the (multiply punctured) plane, i.e., the braid group. The operators which create anyons must therefore be similarly multi-valued and carry this same representation. Such multi-valued operators may be understood as single-valued operators, defined on the universal covering space of the punctured plane, which is the set of homotopy classes of continuous open curves which start at a fixed base point. This dependence on homotopy class cannot appear in the (single-valued) Hamiltonian density and must therefore cancel out through the anyon interactions. As a result, the curve dependence represents a kind of "gauge" freedom,[4] an inevitable feature characteristic of anyon interactions [51]. The appropriate operators have been constructed in the continuum (with its attendant difficulties), and previous work on lattice Maxwell–Chern–Simons theory has centered on generalizing these to the lattice.

To define a multi-valued operator in terms of the fundamental (and single-valued) fields Ψ and A_μ, we shall make use of a multi-valued lattice angle function. Our treatment will be similar to that of Lüscher [32], but with a few differences.

[4] This curve changing "gauge" invariance is not a gauge invariance in the true sense, since the gauge group (i.e., the braid group) need not be represented trivially on the physical Hilbert space. By abuse of language, we here use the term gauge invariance to refer to invariance of the Hamiltonian under a local symmetry.

Given the inverse lattice Laplacian operator $g(x)$: $d \cdot \hat{d} g(x) = -\delta(x)$, defined by (see Ref. [32]),

$$g(x) = \int_{\Omega_B} \frac{d^2p}{(2\pi)^2} \frac{1 - e^{ip \cdot x}}{\sum_i 2(1 - \cos p_i)}, \tag{3.1}$$

let $f_i(x) = \hat{d}_i^{\pm} g(x)$, which then satisfies $-d_i^{\pm} f_i(x) = \delta(x)$. Then the angle function is defined as a contour sum over this vector field from a base point B to the endpoint x along a curve C_x as

$$\theta_{C_x}(x, y) = 2\pi \sum_{l \in C_x} dl_i f_i(l - y). \tag{3.2}$$

C_x is a curve going from the base point B to a point x, $dl_i = \pm \hat{\imath}$ along the directed link going from z to $z \pm \hat{\imath}$, and we have shifted the origin of the vector field f to y. We refer to y as the "origin of the angle function." We choose the base point B so that these curves cannot wind around it, by putting it off at infinity in some direction.

It is easily seen that $\theta_{C_x}(x, y)$ is multi-valued because it increases by 2π when the curve C_x winds around its "center point." The center point of the angle function thusly defined lies on the dual lattice, in the center of the plaquette in whose lower left corner lies the origin of the angle function, the lattice point y. Since a lattice curl—using the forward difference operator d, and acting on a vector function f_i at the point x—amounts to a winding around the plaquette in whose lower left corner lies the point x, we have that this angle function satisfies the two (equivalent) conditions

$$d \times d\theta_C(x, y) = 2\pi\delta(x - y) \tag{3.3a}$$

$$\theta_C(x, y) - \theta_{C'}(x, y) = 2\pi \sum_z \omega(CC'^{-1}, z), \tag{3.3b}$$

where $\omega(CC'^{-1}, z)$ is the winding number of the closed curve CC'^{-1} around the point z and is like the continuum angle function in most respects.[5] There is, however, an important angle function relation which is modified on the lattice. Let $\nu(C_x, C_y)$ be the signed intersection number of C_x with C_y, defined as the number of left-handed intersections minus the number of right-handed intersections (with slight deformations, all curve intersections can be brought to one of these forms). The continuum angle function obeys

$$\theta_{C_x}(x, y) - \theta_{C_y'}(y, x) - \pi \operatorname{sgn}(x_2 - y_2) + 2\pi\nu(C_x, C_y') = 0 \tag{3.4a}$$

[5] We write CC'^{-1} for the contour which is formed by running along first C and then backwards along C'. This notation was chosen to be consistent with that familiar from homotopy theory, and so would be more readily understood. However, as these curves are summation contours, the reader is reminded that this multiplication is to be understood as commutative.

(where $y=(y_1, y_2)$, $x=(x_1, x_2)$); however, the above-defined lattice angle function satisfies (see Appendix)

$$\theta_{C_x}(x, y)-\theta_{C'_y}(y, x)-\pi \operatorname{sgn}(x_2-y_2)+2\pi v(C_x, C'_y)=\xi(x-y) \tag{3.4b}$$

with

$$\xi(x-y)=-\tfrac{1}{2}[f_1(x-y)+f_2(x-y)+f_1(x-y+\hat{2})+f_2(x-y+\hat{1})]; \tag{3.5}$$

ξ satisfies $\xi(x)=-\xi(-x)$.

The properties (3.3), obeyed by the angle function, are unaffected by adding any curve independent nonsingular function to θ. We may take advantage of this fact to alter the definition of the angle function, if we so choose, so as to zero out the right-hand side of (3.4b). The appropriate modification is

$$\tilde{\theta}_{C_x}(x, y)=\theta_{C_x}(x, y)+\tfrac{1}{2}\xi(x-y). \tag{3.6}$$

In what follows, we will need to define θ as a single-valued function, by fixing the curve dependence, i.e., by fixing the "gauge." This is done by choosing a field of curves $\mathscr{C}(x, z)$, depending on both the endpoint x and the origin of the angle function z, which for each x defines a curve running from the base point to the point x. We choose this field of curves so that it is explicitly translation invariant, in the sense that $\mathscr{C}(x, z)=\mathscr{C}(x-z)$. We then define the single-valued function $\theta_{\mathscr{C}}$ as $\theta_{\mathscr{C}}(x-z)=\theta_{\mathscr{C}_{x-z}}(x, z)$. This angle function now has a cut, consistent with the fact that it is no longer multi-valued. For simplicity, we put the origin z of the angle function to zero for the moment. The cut of the angle function may be found as follows: Given two neighboring points x and $x+\hat{\imath}$, and the curves chosen to run to them, C_x and $C_{x+\hat{\imath}}$, define the closed curve $\Delta_i\mathscr{C}_x=C_xL_{x,\hat{\imath}}C_{x+\hat{\imath}}^{-1}$ ($L_{x,\hat{\imath}}$ is the link based at x pointing in the $\hat{\imath}$ direction and may be considered as a curve running from x to $x+\hat{\imath}$) as the curve that comes in from the base point along C_x, continues along the link running from x to $x+\hat{\imath}$, and returns to the base point along the curve $C_{x+\hat{\imath}}$. If this curve winds around the center point of the angle function, then the link from x to $x+\hat{\imath}$ cuts perpendicularly across the cut of the angle function (see Fig. 1). Thus the cut of the angle function does not run along links on the lattice, but between them, on a curve running from the center point of the angle function out to infinity. It is easy to see that $d_i^{\perp}\omega(\Delta_i\mathscr{C}(x), 0)=\delta(x)$, where ω (closed curve, z) is the winding number of a closed curve around the plaquette associated with a point z. In terms of this cut, we have

$$d_i\theta_{\mathscr{C}}(x-z)=2\pi f_i(x-z)+2\pi\omega(\Delta_i\mathscr{C}(x-z), z). \tag{3.7}$$

For future reference, this latter piece satisfies

$$d_i\omega(\Delta_j\mathscr{C}(x-z))-d_j\omega(\Delta_i\mathscr{C}(x-z))=\varepsilon_{ij}\delta(x-z) \tag{3.8a}$$

or

$$d_i^{\perp,x}\omega(\Delta_i\mathscr{C}(x-z), z)=\hat{d}_i^z\omega(\Delta_i^{\perp}\mathscr{C}(x-z), z)=\delta(x-z). \tag{3.8b}$$

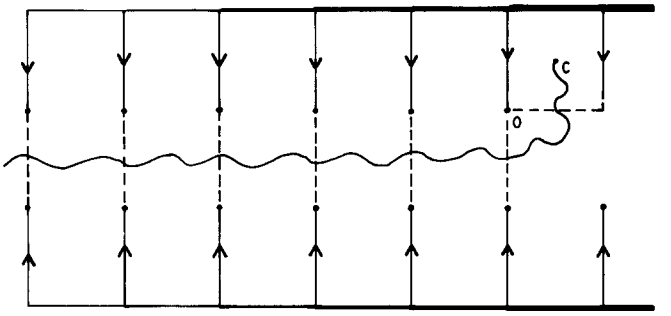

FIG. 1. A curve field $\mathscr{C}$ in the neighborhood of its cut. The wavy line is the cut. The solid lines are curves C in the curve field $\mathscr{C}$. The dotted lines are nonzero links of $\omega(\Delta_i \mathscr{C}(x-z), z)$, which define the cut.

The lattice vector function $\omega(\Delta_i^{\perp}\mathscr{C}(x-z), z)$ may be understood as a "contour density," in that, when dotted into a vector field and summed in the variable z over the entire plane, the result is a contour sum along a lattice curve, with curve parameter z. This curve is a reflection across the origin of the lattice curve beginning at the base point B (i.e., at infinity) and ends at the point x, running next to the cut of the angle function $\theta_{\mathscr{C}}$ (see Fig. 2). The lattice curve defined by $\omega(\Delta_i^{\perp}\mathscr{C}(x-z), z)$ we

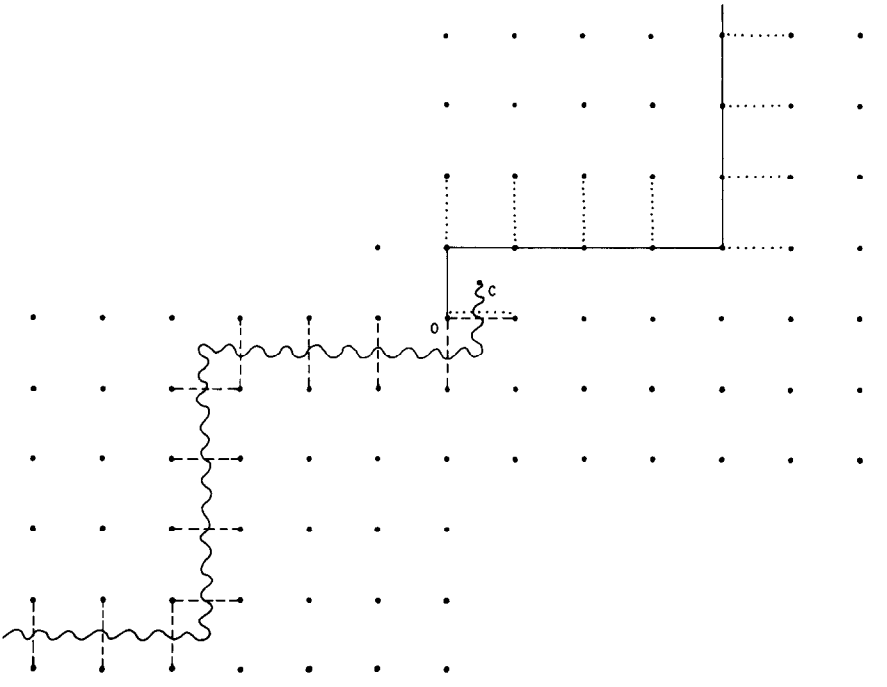

FIG. 2. The cut of a curve field $\mathscr{C}$ and its associated dual cut. The cut is shown as the wavy line, and the dual cut is shown by the solid line. The dotted lines are the reflections of the nonzero links (dashed lines) which define the cut through $\omega(\Delta_i \mathscr{C}(x-z), z)$.

call the "dual cut" $D_{\mathscr{C}}$. Such contours may contain, in addition to the open curve running from B to $x-z$, any number of closed loops, if such loops are contained in the cut of $\theta_{\mathscr{C}}$, by a perverse choice of $\mathscr{C}$.

We will have to consider, in addition to the relations derived for a single angle function above, similar relations for two angle functions defined with different curve fields $\mathscr{C}$ and $\mathscr{C}'$. To this end, we note only that the relation (3.8a) generalizes as follows. Define $\Delta_i(\mathscr{C}, \mathscr{C}'(x)) = C_x L_{x,i} C'^{-1}_{x+\hat{\imath}}$. Then

$$d_i \omega(\Delta_j(\mathscr{C}, \mathscr{C}'(x)), y) - \omega(\Delta_i \mathscr{C}(x+\hat{\jmath}), y) + \omega(\Delta_i \mathscr{C}'(x), y) = \varepsilon_{ij}\delta(x-y). \quad (3.8c)$$

Finally, note that, because $\theta_{\mathscr{C}}$ and f_i are functions of $x-z$ alone, (3.7) implies that $\omega(\Delta_i \mathscr{C}(x-z), z)$ is also a function of $x-z$ alone.

IV. Construction of Anyon Field Operators

In continuum constructions of anyon operators, as well as in the work of Refs. [32, 33], anyon creation operators are constructed by attaching to a fermion field a vortex creation operator $U(x)$, defined as

$$U(x) = \exp i \int d^2z \, d_i \theta(x-z) \, A_i(z). \quad (4.1)$$

This has the effect of making the combination gauge invariant and eliminating the gauge interactions from the Hamiltonian. In addition, a multi-valued field $K_{\mathscr{C}}(x)$ of the form

$$K_{\mathscr{C}} = \exp i \int d^2z \, \theta_{\mathscr{C}}(x, z)(\rho(z) - \rho_0) \quad (4.2)$$

is included whose role is to transmute the statistics. On the lattice the procedure is similar, but we shall have to determine the precise form of these coefficients and $K_{\perp\perp}$ so that anyonization can proceed.

Therefore we write down the anyon creation and annihilation operators as

$$\Phi_{\mathscr{C}}(x) = U(x) \, K_{\mathscr{C}}(x) \, \Psi(x), \qquad \Phi^{\dagger}_{\mathscr{C}}(x) = \Psi^{\dagger}(x) \, K^{\dagger}_{\mathscr{C}}(x) \, U^{\dagger}(x) \quad (4.3)$$

with

$$U(x) = \exp i \sum_z M_i(x-z) \, A_i(z), \qquad K_{\mathscr{C}}(x) = \exp i\alpha \sum_z \theta_{\mathscr{C}}(x-z)(\rho(z) - \rho_0). \quad (4.4)$$

For future reference, we note at this point because of the identity (3.4b) for the angle function, under a change of curve (which shall become a gauge freedom in the anyon Hamiltonian) we have

$$\Phi_{\mathscr{C}'}(x) = \exp i \left(\sum_z \omega(C_x C'^{-1}_x, z)(\rho(z) - \rho_0) \right) \Phi_{\mathscr{C}}(x). \quad (4.5)$$

Consider the Hamiltonian density

$$H = \Psi^\dagger(x+\hat{\imath})\, e^{iA_i(x)}\Psi(x)$$
$$= \Phi^\dagger_{\mathscr{C}}(x+\hat{\imath})\, U(x+\hat{\imath})\, K_{\mathscr{C}}(x+\hat{\imath})\, e^{iA_i(x)} K^\dagger_{\mathscr{C}}(x)\, U^\dagger(x)\, \Phi_{\mathscr{C}}(x). \tag{4.6}$$

We determine the function M_i so that the gauge interaction is eliminated. Actually, unless the statistics parameter is an odd integer, the gauge interaction can only be eliminated at the expense of introducing a charge-dependent Z_N phase, as we will see.

We combine all of the exponential factors accompanying the fields $\Phi_{\mathscr{C}}(x+\hat{\imath})$ and $\Phi_{\mathscr{C}}(x)$ (the Baker–Campbell–Hausdorff commutators give only an irrelevant c-number constant) and examine the phase we obtain. The two $K_{\mathscr{C}}$'s combine to give

$$K_{\mathscr{C}}(x+\hat{\imath})\, K^\dagger_{\mathscr{C}}(x) = \exp i\alpha \sum_z (\rho(z)-\rho_0)\, d_i \theta_{\mathscr{C}}(x-z), \tag{4.7}$$

so that this piece of the phase is now

$$2\pi\alpha \sum_z (\rho(z)-\rho_0)\, \omega(\Delta_i \mathscr{C}(x-z), z) + 2\pi\alpha \sum_z f_i(x-z)(\rho(z)-\rho_0) \equiv \gamma_1 + \gamma_2. \tag{4.8}$$

The first term, γ_1, is the curve dependent interaction which cancels the anyon's curve dependence, as discussed previously. We concentrate on the second term γ_2. We may apply Gauss' law (the other operators present, when taken in certain combinations, are gauge invariant) $B - 2\pi\alpha(\rho-\rho_0) = 0$ and sum by parts to find

$$\gamma_2 = \sum_z d_j^\perp f_i(x-z)\, A_j(z). \tag{4.9}$$

Combining the other three exponentials together, we find, up to an inessential constant,

$$U(x+\hat{\imath})\, e^{iA_i(x)} U^\dagger(x) \equiv e^{i\beta}$$
$$\beta = \sum_z d_i M_j(x-z)\, A_j(z) + A_i(x) \tag{4.10a}$$
$$= \sum_k (d_i M_j(k) + \delta_{ij})\, A_j(-k). \tag{4.10b}$$

Adding, we find

$$\beta + \gamma_2 = \sum_k (d_i M_j(k) + \delta_{ij} + d_j^\perp f_i(k))\, A_j(-k). \tag{4.11}$$

This phase vanishes when

$$M_j = -\frac{\hat{d}_j}{d\cdot\hat{d}} = f_j^\perp(k), \tag{4.12}$$

and our Hamiltonian is simply

$$H = \sum_{x,i} \Phi^{\dagger}_{\mathscr{C}}(x+\hat{\imath})\, U(\Delta_i \mathscr{C}(x))\, \Phi_{\mathscr{C}}(x) \tag{4.13}$$

with

$$\begin{aligned} U(\Delta_i \mathscr{C}(x)) &= e^{i\eta_1} \\ &= \exp i\left(\sum_z \omega(\Delta_i \mathscr{C}(x-z), z)(\rho(z)-\rho_0)\right). \end{aligned} \tag{4.14}$$

We would like to express the sum in the exponential as the charge enclosed inside some curve. In the above form we are close, but the curve itself depends on the summation parameter z. The solution is to make use of the dual cuts defined in Section III. Applying Gauss' law and summing by parts, we have

$$U(\Delta_i \mathscr{C}(x)) = \exp i\left(\sum_z d_j \omega(\Delta_i \mathscr{C}(x-z), z)\, A_j^{\perp}(z)\right) \tag{4.15}$$

which, from (3.8a), is

$$U(\Delta_i \mathscr{C}(x)) = \exp i\left(\sum_z \left[d_i \omega(\Delta_j^{\perp} \mathscr{C}(x-z), z) + \delta_{ij}\delta(x-z)\right] A_j(z)\right). \tag{4.16}$$

This last result may be understood as a difference of Wilson lines on the dual cuts of the angle functions with origins at x and $x+\hat{\imath}$, respectively, plus a piece on the link running from x to $x+\hat{\imath}$, i.e., a closed Wilson loop. Applying Gauss' law one more time, we have

$$U(\Delta_i \mathscr{C}(x)) = \exp i\left(\sum_z \omega(D_{\mathscr{C}}(x+\hat{\imath})\, L_{x,\hat{\imath}} D_{\mathscr{C}}(x)^{-1}, z)(\rho(z)-\rho_0)\right), \tag{4.17}$$

where $D_{\mathscr{C}}(x)$ is the dual cut of the angle function $\theta_{\mathscr{C}}$ with origin x and thus does not change with z. Thus, the phase U has the interpretation of the charge (or flux) enclosed inside the closed curve formed by the dual cuts $D_{\mathscr{C}}(x)$ and $D_{\mathscr{C}}(x+\hat{\imath})$ and the link $L_{x,\hat{\imath}}$.

In general, we may consider fields Φ with different choices of curve field in (4.13) $\mathscr{C}$ and $\mathscr{C}'$. The changes resulting from this in the above calculation are minimal. In Eq. (4.8), the winding number factor that appears in the first term becomes $\omega(\Delta_i(\mathscr{C}, \mathscr{C}')(x-z), z)$, and the generalization of (3.8a) given in (3.8c) is now used in (4.16) to give

$$\begin{aligned} &U(\Delta_i(\mathscr{C}, \mathscr{C}'(x))) \\ &= \exp i\left(\sum_z \left[\omega(\Delta_j^{\perp} \mathscr{C}(x-z+\hat{\imath})) - \omega(\Delta_j^{\perp} \mathscr{C}'(x-z)) + \delta_{ij}\delta(x-z)\right] A_j(z)\right). \end{aligned} \tag{4.18}$$

Again, we see that two dual cut functions, one running from B to x, and the other running from $x+\hat{\imath}$ to B, appear along with a connecting link represented by the $\delta_{ij}\delta(x-z)$ term. Together these are a contour density for a closed curve, so that when multiplied into a gauge field and summed over the plane, they become a Wilson loop, as before. The final result is

$$H=\sum_{x,\hat{\imath}} \Phi^{\dagger}_{\mathscr{C}}(x+\hat{\imath})\, U(\Delta_i(\mathscr{C}, \mathscr{C}'(x)))\, \Phi_{\mathscr{C}'}(x) \tag{4.19}$$

with

$$U(\Delta_i(\mathscr{C}, \mathscr{C}'(x)))=\exp 2\pi i\alpha\left(\sum_z \omega(D_{\mathscr{C}}(x+\hat{\imath})\, L_{x,\hat{\imath}} D_{\mathscr{C}'}(x)^{-1}, z)(\rho(z)-\rho_0)\right). \tag{4.20}$$

Because of the gauge transformation (4.5) under a change of curve, this type of interaction is unavoidable in a theory of anyons [51]. Note that this phase is of the form α times an integer. When the statistics parameter α is an odd integer, this phase is just 1, and, if α is rational, the phase may take on only a finite number of values. In such a case the theory becomes a Z_N "gauge" theory, where N is the rationality of α.

As a final note to this calculation, we point out that it may also be carried out at the level of the Lagrangian. In this case, we have the added interesting phenomenon that the anyonic variables, when plugged into the time derivative part of the Lagrangian (2.1), precisely cancel out the canonical part of the Chern–Simons term, leaving only the constraint times A_0, minus the anyon Hamiltonian (4.13) and a kinetic term for the anyons of the same form as that for fermions, as in (2.1). However, the utility of this is not clear, because of the complicated changes induced in the measure.

We need to impose the condition of anyonic statistics.[6] We thus consider

$$\Phi_{\mathscr{C}}(x)\, \Phi_{\mathscr{C}'}(y)=U(x)\, K_{\mathscr{C}}(x)\, \Psi(x)\, U(y)\, K_{\mathscr{C}'}(y)\, \Psi(y) \tag{4.21}$$

and interchange the order of the exponentials using the Baker–Campbell–Hausdorff theorem $e^A e^B = e^B e^A e^{[A,B]}$. We find

$$\begin{aligned} U(x)\, U(y) &= U(y)\, U(x) \exp -i\pi\alpha \sum_{zz'} M_i(x-z)\, M_j(y-z')\, K^{-1}_{ij}(z-z') \\ &= U(y)\, U(x) \exp -i\pi\alpha \sum_k M_i(k)\, M_j(-k)\, K^{-1}_{ij}(k) \\ &= U(y)\, U(x) \exp -4i\pi\alpha K_{\perp\perp}(x-y), \end{aligned} \tag{4.22}$$

[6] Note that, although it is possible to require the commutation of specific operators, a well-defined anyonic exchange requires that the state in whose presence this operation is performed be empty in order that this operation is meaningful. If it is not, we shall have to require that, in addition to the curve field $\mathscr{C}(x-z)$ avoiding the two charges involved in the commutation, $\mathscr{C}(x-z)$ must avoid all other charges in the plane. Note further that this is not a problem for bosons, only for anyons.

using the commutator function $[A_i(x), A_j(y)] = i\pi\alpha K_{ij}^{-1}(x-y)$ from (2.28). On interchanging $K_{\mathscr{C}}(x)$ with $\Psi(y)$ and $K_{\mathscr{C}}(y)$ with $\Psi(x)$, we obtain the factors

$$\begin{aligned} K_{\mathscr{C}}(x)\,\Psi(y) &= \Psi(y)\,K_{\mathscr{C}}(x)\exp i\alpha\theta_{\mathscr{C}}(x-y), \\ \Psi(x)\,K_{\mathscr{C}}(y) &= K_{\mathscr{C}}(y)\,\Psi(x)\exp -i\alpha\theta_{\mathscr{C}}(y-x), \end{aligned} \tag{4.23}$$

so that

$$\begin{aligned} \Phi_{\mathscr{C}}(x)\,\Phi_{\mathscr{C}}(y) = &-\Phi_{\mathscr{C}}(y)\,\Phi_{\mathscr{C}}(x)\exp i\alpha(\theta_{\mathscr{C}}(x-y)-\theta_{\mathscr{C}}(y-x)) \\ &\times\exp -4i\pi\alpha K_{\perp\perp}(x-y). \end{aligned} \tag{4.24}$$

In the continuum, the difference of angle functions that sits in the first exponential would simply be $\pi\,\mathrm{sgn}(x_2-y_2)-2\pi\nu(C(x-y), C(y-x))$, with $\nu(C(x-y), C(y-x))$ the signed intersection number of the two curves $C(x-y)$ and $C(y-x)$, and this factor alone would give us precisely the anyonic commutation relations. As we have seen, this is not necessarily so for the angle functions that arise naturally on the lattice, most notably the angle function defined in (3.2). Let us imagine for a moment that the previous calculations have been done, not with the angle function of (3.2), but with an arbitrary angle function $\phi_{\mathscr{C}}(x, y)$, which differs from $\theta_{\mathscr{C}}(x, y)$ by the addition of a single-valued function which we call $\frac{1}{2}\chi(x-y)$, and so necessarily still satisfies the two conditions (3.3a), (3.3b). Because they continue to satisfy the two conditions (3.3a), (3.3b), nothing actually changes in the above. Generally, though, we will have a lattice angle deficit:

$$\begin{aligned} &\phi_{\mathscr{C}}(x-y)-\phi_{\mathscr{C}}(y-x)-\pi\,\mathrm{sgn}(x_2-y_2)+2\pi\nu(C(x-y), C(y-x)) \\ &\quad=\tfrac{1}{2}(\chi(x-y)-\chi(y-x)). \end{aligned} \tag{4.25}$$

Aside from the terms $\pi\,\mathrm{sgn}(x_2-y_n)+2\pi\nu(C(x-y), C(y-x))$, this is what then appears in the first exponential of (4.24). It may in general be cancelled off simply by choosing $K_{\perp\perp}$ appropriately. Note that this is possible only because the lattice angle deficit is of necessity an odd function of position, and thus it is imaginary in k-space, as $K_{\perp\perp}$ should be. In particular, for the two lattice angle functions θ and $\tilde{\theta}$ defined in (3.2) and (3.6), respectively, $K_{\perp\perp}$ is chosen as

$$K_{\perp\perp} = \frac{1}{d\cdot\hat{d}}\,\mathrm{Tr}\,K = \frac{\xi}{4\pi} \qquad (\text{for } \theta) \tag{4.26a}$$

$$K_{\perp\perp} = \frac{1}{d\cdot\hat{d}}\,\mathrm{Tr}\,K = 0 \qquad (\text{for } \tilde{\theta}). \tag{4.26b}$$

Thus we see that the formalism above can accommodate any lattice angle function at all, as long as it satisfies (3.3a), (3.3b). Note finally that, for any reasonable lattice angle function, the lattice angle deficit $\frac{1}{2}(\chi(x-y)-\chi(y-x))$ vanishes in the continuum limit, and so it does no damage to the good continuum limit properties of our lattice Chern–Simons term.

We have now shown that there exists an exact lattice anyonization, i.e., a map which eliminates all extra gauge interactions from the Hamiltonian (4.13) (although leaving the necessary curve-changing gauge invariance and achieves the exact anyon commutation relations

$$\Phi_{\mathscr{C}}(x)\,\Phi_{\mathscr{C}}(y) = -\Phi_{\mathscr{C}}(y)\,\Phi_{\mathscr{C}}(x)\exp i\alpha(\pi\,\mathrm{sgn}(x_2-y_2) - 2\pi v[C(x-y), C(y-x)]) \tag{4.27a}$$

$$\Phi^{\dagger}_{\mathscr{C}}(x)\,\Phi^{\dagger}_{\mathscr{C}}(y) = -\Phi^{\dagger}_{\mathscr{C}}(y)\,\Phi^{\dagger}_{\mathscr{C}}(x)\exp -i\alpha(\pi\,\mathrm{sgn}(x_2-y_2) - 2\pi v[C(x-y), C(y-x)]) \tag{4.27b}$$

$$\Phi_{\mathscr{C}}(x)\,\Phi^{\dagger}_{\mathscr{C}}(y) = -\Phi^{\dagger}_{\mathscr{C}}(y)\,\Phi_{\mathscr{C}}(x)\exp i\alpha(\pi\,\mathrm{sgn}(x_2-y_2) - 2\pi v[C(x-y), C(y-x)]), \qquad x\neq y \tag{4.27c}$$

V. Wilson Line Formalism

Although the foregoing is the traditional formalism in which to treat anyonization, a much simpler formalism is in fact available. We return to the definition of our anyon variables and apply Gauss' to $K_{\mathscr{C}}$,

$$\Phi_{\mathscr{C}}(x) = \exp i\sum_z f_i^{\perp}(x-z)\,A_i(z)\exp i\alpha\sum_z \theta_{\mathscr{C}}(x-z)(\rho(z)-\rho_0)$$

$$= \exp i\sum_z f_i^{\perp}(x-z)\,A_i(z)\exp -i\frac{\alpha}{2\pi}\sum_z \theta_{\mathscr{C}}(x-z)\,d_i^{\perp}\,A_i(z). \tag{5.1}$$

We may now sum by parts freely, because $\theta_{\mathscr{C}}$ is a single-valued function. From (3.7), we will have two terms. The first simply cancels out the first exponential factor. The other term is $\omega(\Delta_i^{\perp}\mathscr{C}(x-z), z)$, which satisfies (3.8b) and is, therefore, a Wilson line. Thus we see that the effect of all of the above machinery is to attach a Wilson line to this fermion field, with *no* extra gauge invariant phase (unless such phases are added by a perverse choice of $\mathscr{C}$, which may add other Wilson loops to the Wilson line). Note that this result depends on the particular choice of angle function that we have made, namely that defined in (3.2).

We may then redo the calculations for the elimination of the gauge interaction and anyonic statistics very simply. In fact, the first calculation is trivial. With

$$\Phi_{\mathscr{C}}(x) = \Psi(x)\,W(D_{\mathscr{C}}(x)), \qquad W(D_{\mathscr{C}}(x)) = \exp i\sum_{l\in D_{\mathscr{C}}(x)} dl_i\,A_i(l), \tag{5.2}$$

we find that $\Psi^{\dagger}(x+\hat{\imath})\exp iA_i(x)\,\Psi(x)$ becomes

$$\Psi^{\dagger}(x+\hat{\imath})\exp iA_i(x)\,\Psi(x) = \Phi^{\dagger}_{\mathscr{C}}(x+\hat{\imath})\,W(D_{\mathscr{C}}(x+\hat{\imath}))\,e^{iA_i(x)}W^{\dagger}(D_{\mathscr{C}}(x))\,\Phi_{\mathscr{C}}(x). \tag{5.3}$$

It is immediate that the middle three factors together form a Wilson loop,

$$W(D_{\mathscr{C}}(x)\, L_{x,\hat{\imath}} D_{\mathscr{C}}^{-1}(x+\hat{\imath})) = \exp i \sum_{l \in D_{\mathscr{C}}(x) L_{x,\hat{\imath}} D_{\mathscr{C}}^{-1}(x+\hat{\imath})} A_i(l)\, dl_i$$

$$= \exp i \sum_z B(z)\, \omega(D_{\mathscr{C}}(x)\, L_{x,\hat{\imath}} D_{\mathscr{C}}^{-1}(x+\hat{\imath}), z)$$

$$= \exp 2\pi i\alpha \sum_z (\rho(z) - \rho_0)\, \omega(D_{\mathscr{C}}(x)\, L_{x,\hat{\imath}} D_{\mathscr{C}}^{-1}(x+\hat{\imath}), z) \tag{5.4}$$

which is precisely the factor $U(\Delta_i(\mathscr{C}, \mathscr{C}'(x)))$ of (4.19) and (4.20).

To compute the statistics of these $\Phi_{\mathscr{C}}$'s, we need only the commutator of the Wilson lines,

$$\Phi_{\mathscr{C}}(x)\, \Phi_{\mathscr{C}}(y) = -e^{\Gamma} \Phi_{\mathscr{C}}(y)\, \Phi_{\mathscr{C}}(x) \tag{5.5}$$

$$\Gamma = -\left[\sum_{l \in D_{\mathscr{C}}(x)} dl_i A_i(l), \sum_{l' \in D_{\mathscr{C}}(y)} dl'_j A_j(l') \right]$$

$$= -i\pi\alpha \sum_{l \in D_{\mathscr{C}}(x)} \sum_{l' \in D_{\mathscr{C}}(y)} dl_i\, dl'_j K_{ij}^{-1}(l - l')$$

$$= -2\pi i\alpha \sum_{l \in D_{\mathscr{C}}(x)} \sum_{l' \in D_{\mathscr{C}}(y)} dl_i\, dl'_j \left\{ \hat{d}_i^{\perp} \hat{d}_j \left(l \left| \frac{1}{d \cdot \hat{d}} \right| l' \right) - d_i d_j^{\perp} \left(l \left| \frac{1}{d \cdot \hat{d}} \right| l' \right) \right\}$$

$$-i\pi\alpha \frac{1}{\pi} \sum_{l \in D_{\mathscr{C}}(x)} \sum_{l' \in D_{\mathscr{C}}(y)} dl_i\, dl'_j\, d_i \hat{d}_j \xi(l - l'). \tag{5.6}$$

Allowing the hatted differences of $l - l'$ to act as unhatted differences of l', we may evaluate four of these six contour sums directly:

$$\Gamma = -2\pi i\alpha \left[\sum_{l \in D\xi_{\mathscr{C}}(x)} dl_i \hat{d}_i^{\perp} \left(l \left| \frac{1}{d \cdot \hat{d}} \right| y \right) - \sum_{l' \in D_{\mathscr{C}}(y)} dl'_j d_j^{\perp} \left(x \left| \frac{1}{d \cdot \hat{d}} \right| l' \right) \right]$$

$$- i\alpha\xi(x - y). \tag{5.7}$$

The first term we recognize as $\theta_{D_{\mathscr{C}}(x)}(x, y)$. The second term we rewrite as

$$2\pi i\alpha \sum_{l \in D_{\mathscr{C}}(y)} dl_i \hat{d}_i^{\perp} \left(l \left| \frac{1}{d \cdot \hat{d}} \right| x \right) \tag{5.8}$$

and this we recognize as $\theta_{D_{\mathscr{C}}(y)}(y, x)$. The final result for Γ is

$$\Gamma = i\alpha(\theta_{D_{\mathscr{C}}(x)}(x, y) - \theta_{D_{\mathscr{C}}(y)}(y, x) - \xi(x - y))$$

$$= i\pi\alpha\, \mathrm{sgn}(x_2 - y_2) - 2\pi i\alpha v(D_{\mathscr{C}}(x), D_{\mathscr{C}}(y))), \tag{5.9}$$

and we recover the anyonic statistics property of the field operators $\Phi_{\mathscr{C}}(x)$.

An interesting consequence of (5.9) is that, because (given the choices (3.2) for the angle function and (4.26a) for $K_{\perp\perp}$) commutation of Wilson lines counts their crossings, the commutator function $K_{ij}^{-1}(x - y)$ must be a local function, in the

sense that a link variable commutes nontrivially only with its nearest neighbors. This may seem strange, given (2.28) and (4.26a), but it may be verified directly through tedious algebra. We find

$$K_{ij} = \begin{pmatrix} S_2 - S_2^{-1} & -(-1 + S_2^{-1} + S_1 + S_2^{-1}S_1) \\ -1 + S_1^{-1} + S_2 + S_1^{-1}S_2 & S_1^{-1} - S_1 \end{pmatrix}$$
$$= \begin{pmatrix} d_2 + \hat{d}_2 & -2 - 2d_1 + 2\hat{d}_2 + \hat{d}_2 d_1 \\ 2 + 2d_2 - 2\hat{d}_1 - \hat{d}_1 d_2 & -d_1 - \hat{d}_1 \end{pmatrix} \tag{5.10}$$

which, it may be checked, satisfies both (2.19b) with $J = 1$ and (4.26a). Note that $A_i(x)$ commutes nontrivially with *all* six of its nearest neighbors, i.e., any link with a common site, not just the other link defined to sit at the same site x, as many be read off from the inverse of (5.10) (or see Fig. 3). This appears to restore the "democracy" between the forward and backward directions, first broken by our link variable definitions, but the signs in (5.10) still distinguish these directions. The above result may also be deduced by use of (5.9) along with $A_i(x) = V(D_{\mathscr{C}}(x + \hat{\imath})) - V(D_{\mathscr{C}})(x)$, where $V(D_{\mathscr{C}})(x) = \sum_{l \in D_{\mathscr{C}}(x)} dl_i A_i(l)$.

Armed with this insight, we now return to (5.9). This equation has the odd feature that, while it implies that commutation is local in the above sense, it seems to have an additional feature that does not look local, namely, the first term on the right-hand side, the π term. Although it looks very much like this should be called a "half intersection" and attributed to the common base point of the two contours, this question has never been resolved in the previous, continuum formulations [8, 9, 12]. Because our theory is well regulated, however, we may compute the commutator of Wilson lines intersecting at their endpoints. A trivial computation directly verifies the above conjecture.

This definition of K_{ij}, which makes Wilson line commutators local, allows a geometric formulation of lattice Chern–Simons theory as the geometry of a line bundle over the lattice, as in the continuum. The role of K_{ij} is to define the wedge product so that the Leibniz rule is satisfied. This will be more fully developed elsewhere [53].

We close this section with a remark about the relation of anyons to knot theory. At the beginning of Section III, it was noted that anyon creation operators must

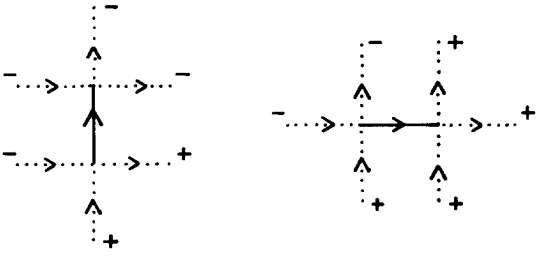

FIG. 3. The link variables which commute nontrivially (dashed lines) with a given link (solid line) are shown, together with the signs of their (c-number) commutators. The magnitudes of these numbers are all 1.

carry a representation of the braid group. It is easy to check that this is in fact the case, because of a theorem by Reidemeister [56–58], which gives necessary and sufficient conditions for a set of permutation generators σ to represent the braid group. Let σ_{ij} represent a permutation of the ith and jth strands. Then Reidemeister requires

$$\begin{aligned} &\text{(I)} \quad \sigma_{ij}\sigma_{kl} = \sigma_{kl}\sigma_{ij}, \qquad i, j \neq k, l, \\ &\text{(II)} \quad \sigma_{ij}\sigma_{jk}\sigma_{ij} = \sigma_{jk}\sigma_{ij}\sigma_{jk}. \end{aligned} \tag{5.11}$$

It is trivial to check that, for a multi-particle state written as a series of anyon creation operators acting on the vacuum, these relations are satisfied by simply commuting creation operators with each other using (4.27).

A surprising aspect of this relationship is that the knot invariances (5.11) are those of framed knots, i.e., knots which have a width and thus are actually ribbons. Unframed invariance requires that the generators σ also not distinguish twists, which are not even defined on these particle states. Thus, the appropriate invariance for anyons is the framed invariance, also known as ambient isotopy, rather than the unframed one, known as regular isotopy. The origins of this framing are not obvious, because the curves involved are merely summation contours, not ribbons. It is worth noting, however, that the forward differencing convention that we introduced in Section I may be regarded as fixing a framing by associating a plaquette to each link and this framing cannot twist. In Section VII, we discuss another lattice Chern–Simons term in which framing of anyon creation operators is more explicit and, also, nontrivial.

VI. Relations to Spin Models and Superfluidity

In the special case that α is an even integer, we have a system of free fermions, and the fields are curve independent. In the case that α is an odd integer, Φ_C and $\Phi_{C'}$ are bosonic and again independent of C and C'. With the identification found in Ref. [48] (see also Ref. [43], $\Phi(x) = S^1(x) - iS^2(x)$, $\Phi^\dagger = S^1(x) + iS^2(x)$, $\rho(x) = S^3(x) - \frac{1}{2}$), they form the $j = \frac{1}{2}$ representation of the lattice SU(2) algebra

$$[S^a(x), S^b(y)] = i\varepsilon^{abc} S^c(x)\,\delta(x-y), \qquad \sum_{a=1}^{3} S^a(x)\,S^a(x) = \tfrac{3}{4}, \tag{6.1}$$

and (11) with $\Delta = -\frac{1}{2}$ is the Hamiltonian of the X–Y model,[7]

$$H = -\sum_{x,i} (S^1(x+i)\,S^1(x) + S^2(x+i)\,S^2(x)). \tag{6.2}$$

[7] The discrete symmetries of Chern–Simons theory map onto the XY model as follows: $(CP)_{CS} \leftrightarrow (CP)_{XY}$ and $(T)_{CS} \leftrightarrow (T)_{XY}$. Parity, which is broken in general for Chern–Simons theory, is not broken in the case that α is an integer.

Here, the average charge density $\rho_0 = \langle Q \rangle$/volume, where Q is the number operator, also fixes the expectation value $\langle \sum_x S^3(x)/V \rangle = \rho_0 - \frac{1}{2}$. We consider the case $\rho_0 = \frac{1}{2}$. It has been proved [49, 50] that, when the volume is infinite, the ground state of the X–Y model in two dimensions has long-range order with $\lim_{x \to \infty} \langle S^1(x)\, S^1(0) \rangle \neq 0$. Thus, in the Chern–Simons–fermion model described by (2.1) and (2.5) (together with the constraints we have derived), the ground state has off-diagonal long-range order,

$$\lim_{x \to \infty} \left\langle \psi^\dagger(x) \prod_{\text{links}} e^{iA_i} \psi(0) \right\rangle \neq 0; \tag{6.3}$$

i.e., it is a charged superfluid and, if coupled to the physical electromagnetic field, at least for sufficiently weak coupling, would be a superconductor. It is also known [50] that the long-range order persists in the antiferromagnetic XXZ model formed by adding the coupling $H_{33} = g \sum_{x,i} S^3(x+i)\, S^3(x)$ or, in fermionic variables,

$$H_{33} = g \sum_{x,i} (\psi^\dagger(x+i)\, \psi(x+i)\, \psi^\dagger(x)\, \psi(x)) - 2gQ + \text{const.}, \tag{6.4}$$

with g sufficiently small. This is a repulsive interaction and, for g large enough, it destroys the superfluid condensate. For $g \to \infty$ the groud state of $H + H_{33}$ is that of H_{33} which is the antiferromagnetic Ising model. The ground state of the latter does not have off-diagonal long-range order and is not a superfluid but breaks symmetry under translation by one lattice spacing, $\langle \sum_x (-1)^{\sum_i x_i} S^3(x) \rangle = \langle \sum_x (-1)^{\sum_i x_i} (\rho(x) - \frac{1}{2}) \rangle \neq 0$. Here, antiferromagnetism of the XXZ model coincides with a commensurate charge-density wave state of the Chern–Simons–fermion model and is stable when the repulsive self-interaction of the fermions is strong enough.

The treatment by Kennedy, Lieb, and Shastry actually may be used to say some things about lattice Chern–Simons theory with more general statistics parameters. Their strategy was to derive an infrared ($q \neq 0$) bound on the correlator $g(q) = \langle S^1_q S^1_{-q} \rangle \leqslant f^{ir}(q)$ as a function of q then integrate it up against $\sum_i \cos q_i$ over the Brillouin zone, thus converting it into a bound on the ground state energy. They then show that, if g is smooth at $q = 0$, i.e., having no δ-function contribution there, then this bound on the energy is contradicted by the simplest of variational bounds, in which the variational state is chosen as an eigenstate of S^1_x at every site x, giving energy per bond $e = -\frac{1}{4}$.

The analogous bound on the correlator of Chern–Simons theory at non-integer statistics parameter is a challenging problem. The KLS proof does not easily carry over to the more general case. But while we cannot complete a proof of long-range order using a straightforward generalization of the KLS strategy, we can use that strategy to find a condition for long-range order in these systems, a condition which may be tested numerically. Because the Hamiltonian of the anyon system is, like that of the XY model, a hopping Hamiltonian, we may still integrate the two-point correlator up against $\cos q_i$ to obtain the ground state energy. Thus an upper

bound, *any* upper bound, on the correlator at finite q may be converted in this way into an upper bound on the ground state energy. If this contradicts some variational bound, we then once again have found long-range order. Variational bounds are more difficult to evaluate with a Hamiltonian such as (4.13), which contains nonlocal phases, but it is possible to show that the state

$$|\Psi\rangle_{\text{var}} = \prod_x{}' \frac{1}{\sqrt{2}} (|1\rangle_x |0\rangle_{x+\hat{2}} + |0\rangle_x |1\rangle_{x+\hat{2}}) \tag{6.5}$$

has energy $-\frac{1}{8}$ per bond, where sites x have components $x = (x_1, x_2)$, and the product $\prod'$ runs over all x_2 even. This variational state may be thought of as a state that "resonates" in the $\hat{2}$ direction. In this configuration, it is possible to choose the curves so that they contain no charge at all, so that the energy is the same as a similar configuration of bosons.

This condition is numerically useful, because it requires the measurement of only a single quantity, an integral of the correlator, and allows an inference of the long-range properties of the system from those at finite distance, which are easier to compute accurately in a numerical study. Computation of the two-point function in the ground state may be treated by standard methods at zero temperature, for example, by using the Lanczos method to find the ground state and evaluating the two-point function straightforwardly. Thus, we may used the above observations as a numerical signal for the onset of anyon superfluidity. We emphasize that this condition is sufficient but not necessary. A numerical search for a KLS-like bound among more exotic spin models has been carried out by Liang [59].

VII. An Exotic Lattice Chern–Simons Theory: Dumbbells

Thus far, we have seen that it is possible to perform an anyonization transformation on a Chern–Simons system written on a lattice. The result was complicated to write down, though easy to understand—while our lattice variables imposed a preferred direction (forward vs backward) on the lattice through the definitions of the link variables defined to be at the site x, the system partially restores the democracy between the positive and negative directions on the lattice, so as to count crossings of Wilson lines correctly. We saw also that this preference could be interpreted as a framing on the Wilson line defining the anyonic variables, thus explaining the peculiarity that anyonic variables seem to obey framed link invariants, rather than unframed ones. All of this suggests that an ordinary square lattice is not the natural setting for lattice Chern–Simons theory. It would be interesting to find a lattice on which Chern–Simons theory sits more naturally, as this would simplify computations and allow the knot-theoretic properties to be more readily extracted.

Recently, Kantor and Susskind [54] proposed a representation of lattice Chern–Simons theory which seems to achieve this program, but involves twice as

many lattice degrees of freedom, defined on a lattice together with its dual lattice. Using a dual lattice in this way to define natural geometric invariants on lattice principal fibre bundles was investigated earlier by Honan [55]. In this section we investigate the properties of the model, constructing exact anyon operators, calculating their monodromy, and treating the continuum limit.

Kantor and Susskind's model introduces a discretized Chern–Simons theory with two species of gauge fields, one associated with links of the primary lattice and the other with links of the dual lattice. In terms of these, they wrote down a very simple, gauge invariant (mixed) term that is completely local. They also suggested that this lattice Chern–Simons theory be coupled to "dumbbell" matter, the components of which occupy sites on the primary lattice as well as an adjacent site of the dual lattice.

We treat the model in the context of the methods developed in Section V. We begin by showing that the Chern–Simons term in that model is parity invariant. We then consider the coupling of the gauge fields to matter fields and show that this, in fact, breaks parity. We further show that the physical states of the theory exhibit anyonic monodromy in terms of a particular version of the two-color braid group of R^2. Using the formalism already developed, we then provide a simple, exact construction of the anyonic fields, which eliminates the need for the semiclassical arguments of Kantor and Susskind. We argue that the lattice Chern–Simons theory which is obtained can be regarded as a lattice regularization of a parity invariant continuum $U(1)$ gauge theory with two species of gauge fields and a parity invariant Chern–Simons term, together with a parity-breaking constraint on the currents.

Consider a square two-dimensional lattice, with sites $x = n_1\hat{1} + n_2\hat{2}$ and basis vectors $\mu = \pm\hat{1}, \pm\hat{2}$, which is called the primary lattice. An oriented link from site x to site $x+\mu$ is denoted $[x, \mu]$. Similarly we consider the dual lattice with sites x^* located at the centers of plaquettes of the primary lattice. We take the convention that the link $[x, \mu]^*$ dual to $[x, \mu]$ is that link of the dual lattice which crosses $[x, \mu]$ and, if $[x, \mu]$ is oriented in the forward direction, $[x, \mu]^*$ is oriented from right-to-left. Then we introduce gauge fields $A([x, \mu], t)$ associated with the link $[x, \mu]$, $\tilde{A}([x, \mu]^*, t)$ associated with $[x, \mu]^*$. $A(x, t)$ is associated with sites, and $\tilde{A}(y^*, t)$ is associated with dual sites. These have the gauge transformations

$$A([x, \mu], t) \to A([x, \mu], t) + d_\mu \chi(x, t), \qquad A(x, t) \to A(x, t) + \dot{\chi}(x, t) \tag{7.1}$$

$$\tilde{A}([x, \mu]^*, t) \to \tilde{A}([x, \mu]^*, t) + \tilde{d}_{\tilde{\mu}} \tilde{\chi}(x^*, t), \qquad \tilde{A}(y^*, t) \to \tilde{A}(y^*, t) + \dot{\tilde{\chi}}(y^*, t), \tag{7.2}$$

where $\chi(x, t)$ and $\tilde{\chi}(y^*, t)$ are independent functions on the primary lattice and dual lattice, respectively and the dual lattice derivative $\tilde{d}_{\tilde{\mu}} \tilde{\chi}(y^*)$ is defined by the operation of taking the difference of $\tilde{\chi}(y^*)$ at the dual sites on the endpoints of $[y, \mu]^*$. Gauge fields have the property

$$A([x, -\mu], t) = -A([x-\mu, \mu], t), \qquad \tilde{A}([x, -\mu]^*, t) = -\tilde{A}([x-\mu, \mu]^*, t). \tag{7.3}$$

The primary lattice gauge field has magnetic flux through a plaquette, i.e., a site of the dual lattice,

$$B(x^* = x + \hat{1}/2 + \hat{2}/2, t) = A([x+\hat{1}, \hat{2}], t) - A([x, \hat{2}], t) - A([x+\hat{2}, \hat{1}], t) + A([x, \hat{1}], t). \quad (7.4)$$

Similarly, the magnetic flux due to the dual lattice field is located on a primary latice site,

$$\tilde{B}(x) = \tilde{A}([x, \hat{1}]^*, t) + \tilde{A}([x, \hat{2}]^*, t) - \tilde{A}([x-\hat{1}, \hat{1}]^*, t) - \tilde{A}([x-\hat{2}, \hat{2}]^*, t). \quad (7.5)$$

A Chern–Simons term which is invariant under the both gauge transformations (7.1) and (7.2), when the gauge functions have compact support, is [54]

$$S_{CS} = \frac{1}{\theta} \int dt \left(\sum_{x, \mu = \hat{1}, \hat{2}} A([x, \mu], t) \frac{d}{dt} \tilde{A}([x, \mu]^*, t) + \sum_x A(x, t) \tilde{B}(x, t) - \sum_{x^*} \tilde{A}(x^*, t) B(x^*, t) \right). \quad (7.6)$$

Canonical quantization of the first-order action (7.6) yields the canonical commutation relation

$$[A([x, \mu]), \tilde{A}([y, \nu]^*)] = i\theta\delta([x, \mu], [y, \nu]). \quad (7.7)$$

In two dimensions, parity is defined as reflecting one of the coordinates, $(x_1', x_2') = (-x_1, x_2)$. Choosing A to be a vector under parity and $\tilde{A}$ a pseudo-vector,

$$A'([x, \hat{1}], t) = A([x', -\hat{1}], t), \qquad A'([x, \hat{2}], t) = A([x', \hat{2}], t) \quad (7.8)$$

$$\tilde{A}'([x, \hat{1}]^*, t) = -\tilde{A}([x', -\hat{1}]^*, t), \qquad \tilde{A}'([x, \hat{2}], t) = -\tilde{A}([x', 2]^*, t) \quad (7.9)$$

$$A'(x, t) = A(x', t), \qquad \tilde{A}'(x^*, t) = -\tilde{A}(x'^*, t), \quad (7.10)$$

we see that this Chern–Simons action is actually invariant. The action (7.6) is also invariant under the simultaneous charge conjugation transformations $A \to -A$, $\tilde{A} \to -\tilde{A}$. The continuum limit of (7.6) is

$$S_{SC} = \frac{1}{\theta} \int dt\, d^2x \left(\varepsilon_{ij} A_i(x) \frac{d}{dt} \tilde{A}_j(x) + A_0(x) \tilde{B}(x) - \tilde{A}_0(x) B(x) \right), \quad (7.11)$$

where, again, A is a vector under parity and $\tilde{A}_\mu$ is a pseudo-vector, so that the Chern–Simons cross-coupling is parity invariant. Such a model has been used to discuss a scenario for parity symmetric anyon superconductivity [29].

This gauge theory is suited to couple to matter which resides either on the primary lattice or the dual lattice and which has transport confined to its respective lattice. Gauge invariant transport of a particle with charge g along an oriented

curve C on the primary lattice or a particle with charge $\tilde{g}$ along an oriented curve $\tilde{C}$ of the dual lattice are accompanied by the Wilson line operators,

$$W[C] = \exp ig \sum_{\text{links in } C} A([x, \mu]), \qquad \tilde{W}[\tilde{C}] = \exp i\tilde{g} \sum_{\text{dual links in } \tilde{C}} \tilde{A}([x, \mu]^*). \quad (7.12)$$

If x_i, x_f are the initial and final endpoints of C and x_i^*, x_f^* the initial and final points of $\tilde{C}$, the operators in (12) gauge transform as

$$W[C] \to e^{ig\chi(x_f)} W[C]\, e^{-ig\chi(x_i)}, \qquad \tilde{W}[\tilde{C}] \to e^{i\tilde{g}\tilde{\chi}(x_f^*)} \tilde{W}[\tilde{C}]\, e^{-i\tilde{g}\tilde{\chi}(x_i)}. \quad (7.13)$$

Also, for closed curves, they are gauge invariant and give the monodromy

$$\begin{aligned} W[\text{closed } C] &= \exp ig \sum_{x^* \in \text{interior } C} B(x^*), \\ \tilde{W}[\text{closed } \tilde{C}] &= \exp i\tilde{g} \sum_{x \in \text{interior } \tilde{C}} \tilde{B}(x) \end{aligned} \quad (7.14)$$

and for either open or closed curves they have the commutator algebra,

$$W[C]\, W[C'] = W[C']\, W[C], \qquad \tilde{W}[\tilde{C}]\, \tilde{W}[\tilde{C}'] = \tilde{W}[\tilde{C}']\, \tilde{W}[\tilde{C}] \quad (7.15)$$

$$\tilde{W}[\tilde{C}]\, W[C] = W[C]\, \tilde{W}[\tilde{C}] \exp ig\tilde{g}\theta v[\tilde{C}, C], \quad (7.16)$$

where $v[\tilde{C}, C]$ is the number of right-handed intersections minus the number of left-handed intersections of $\tilde{C}$ and C (this is strictly true only when C and $\tilde{C}$ do not intersect each other's endpoints). It follows that for closed loops C and $\tilde{C}$, $W[C]$ and $\tilde{W}[\tilde{C}]$ always commute. It further follows that simple counterclockwise non-tilde (tilde) Wilson loops acquire a phase $\exp ig\tilde{g}\theta$ ($\exp -ig\tilde{g}\theta$) when commuted with a tilde (non-tilde) Wilson line whose endpoint is inside the loop. More generally, the commutation phase between a Wilson line and a Wilson loop is proportional to the winding number of the loop around the endpoint of the Wilson line.

We introduce annihilation and creation a, $a^\dagger$ for particles with charge g which reside on primary lattice sites and $\tilde{a}$, $\tilde{a}^\dagger$ for particles with charge $\tilde{g}$ which reside on dual sites with nonvanishing anti-commutators

$$\{a(x), a^\dagger(y)\} = \delta_{xy}, \qquad \{\tilde{a}(x^*), \tilde{a}^\dagger(y^*)\} = \delta_{x^*y^*}. \quad (7.17)$$

Their gauge transformations are $a(x) \to \exp + i\Lambda(x)\, a(x)$, $a^\dagger(x) \to \exp - i\Lambda(x)\, a^\dagger(x)$, $\tilde{a}(x) \to \exp + i\tilde{\Lambda}(x)\, \tilde{a}(x)$, $\tilde{a}^\dagger(x) \to \exp - i\tilde{\Lambda}(x)\, \tilde{a}^\dagger(x)$. The generators of gauge transformations are

$$\begin{aligned} \mathscr{G}_\chi &= \sum_x \chi(x) \left(g \frac{1}{2} [a^\dagger(x), a(x)] + \frac{1}{\theta} \tilde{B}(x) \right), \\ \mathscr{G}_{\tilde{\chi}} &= \sum_{x^*} \tilde{\chi}(x^*) \left(\tilde{g} \frac{1}{2} [\tilde{a}^\dagger(x^*), \tilde{a}(x^*)] + \frac{1}{\theta} B(x^*) \right). \end{aligned} \quad (7.18)$$

Here we have defined the matter charge densities using a Dirac commutator. In a relativistic theory or a non-relativistic theory which has a particle–hole symmetry, this definition gives the gauge generators a definite charge conjugation symmetry, where the matter transformation law is $C: a(x) \to a^\dagger(x)$, $a^\dagger(x) \to a(x)$ and $C: \tilde{a}(x^*) \to \tilde{a}^\dagger(x^*)$, $\tilde{a}^\dagger(x^*) \to \tilde{a}(x^*)$. (This is an invariance of the anti-commutators in (7.17). It generally has to be augmented by a gauge transformation to also be an invariant transformation of a Hamiltonian.) Furthermore, under parity $P: a, a^\dagger(x) \to a, a^\dagger(x')$ and $P: \tilde{a}, \tilde{a}^\dagger(x^*) \to \tilde{a}^\dagger, \tilde{a}(x^{*\prime})$. Note that the parity transformation of $\tilde{a}$ is defined so that $\tilde{\mathscr{G}}$ in (7.18) has well-defined parity in that both terms in the summand are pseudo-scalars.

Gauge invariance of a physical model with charged particles requires the constraints $\mathscr{G} \sim 0$ and $\tilde{\mathscr{G}} \sim 0$ which are imposed as physical state conditions. Gauge invariant (under transformations where the gauge functions have compact support) creation and annihilation operators are formed by taking a product of a creation or annihilation operator at a point x or x^* and the appropriate Wilson line operator with curve C or $\tilde{C}$ going from the point x or x^* to some point at infinity,

$$\alpha_C(x) = W[C]\, a(x), \qquad \tilde{\alpha}_{\tilde{C}}(x^*) = \tilde{W}[\tilde{C}]\, a(x^*). \tag{7.19}$$

These are multi-valued operators with multi-valuedness characterized by the monodromy in (7.14). Thus, for example, when we transport the operator $\alpha_C(x)$ around a loop C' we obtain

$$\begin{aligned} \alpha_{C+C'}(x) &= \exp\left(ig \sum_{x^* \in \text{interior } C'} B(x^*) \right) \alpha_C(x) \\ &\sim \exp\left(ig\tilde{g}\theta \sum_{x^* \in \text{interior } C'} \tilde{a}^\dagger(x^*)\, \tilde{a}(x^*) \right) \alpha_C(x), \end{aligned} \tag{7.20}$$

where the last weak equality holds for matrix elements with physical states. Thus the change in phase of the wavefunction upon transport of an α-particle around a loop is given by $\theta g\tilde{g}$ times the number of $\tilde{\alpha}$-particles whose positions are linked by the loop. If $\theta g\tilde{g}/2\pi$ is a rational number (say M/N, where M and N are relative prime integers) this monodromy can only take on a finite number of values which are the elements of the one-dimensional representation of the discrete group Z_N by phases $\exp 2\pi i Mn/N$. In the following we shall confine our attention to this case.

We quantize the gauge fields using the functional Schroedinger picture. We treat the primary gauge field $A([x, \mu])$ as a coordinate, so that the wavefunctionals depend on classical configurations of $A([x, \mu])$ and we represent the commutator (7.7) by taking

$$\tilde{A}([x, \mu]^*) \equiv -i\theta \frac{\delta}{\delta A([x, \mu])}. \tag{7.21}$$

Then the empty vacuum state is given by[8]

$$\Psi_0[A] = |0\rangle \prod_{x^*} \delta(B(x^*)), \tag{7.22}$$

where $|0\rangle$ is the state with no charged particles $a(x)\,|0\rangle = 0 = \tilde{a}(x^*)\,|0\rangle$. From the empty vacuum we can create the basis states for the n-particle and $\tilde{n}$-particle sector of the theory,

$$\alpha^\dagger_{C_1}(x_1)\cdots\alpha^\dagger_{C_n}(x_n)\,\tilde{a}^\dagger_{\tilde{C}_1}(x_1^*)\cdots\tilde{a}^\dagger_{\tilde{C}_{\tilde{n}}}(x_{\tilde{n}}^*)\,\Psi_0[A]. \tag{7.23}$$

In this state, we transport a particle in a loop, l, by successive operation from the left with the gauge invariant operator $T([x,\mu]) = a^\dagger(x+\mu)\,e^{iA([x,\mu])}a(x)$. This effectively multiplies the state from the left by the Wilson loop operator $W[l]$. We can use lattice Stoke's theorem to show that $W[l]\,\Psi_0 = \Psi_0$, so that the only effect of this transport resides in the commutators of $W[l]$ with $\tilde{W}[\tilde{C}_i]$. From the discussion after Eq. (7.16) it is easy to see that the net phase is $\exp(ig\tilde{g}\theta \times$ (number of tilde particles linked by l)), because each of these particles sits at the end of a Wilson line. Also, if the non-tilde particle at x_1 is exchanged with the non-tilde particle at x_2 by transporting particle 1 along C_{12} and particle 2 along C_{21} then the wavefunction acquires a phase $\exp(ig\tilde{g}\theta$ (number of tilde-particles linked by) $C_{12} \oplus C_{21})$. For similar reasons, it can be shown that if we transport a tilde-particle in a loop or exchange two tilde-particles the state changes by a phase $\exp(ig\tilde{g}\theta$ (number of non-tilde-particles linked by the loop)). (Note that the sign of the phase is the same as that for the non-tilde particle, so that if a pair of bound states of two such particles were to encircle each other, these phases would add.) Thus the wavefunction carries a one-dimensional unitary representation of a two-color braid group [57]. The representation gives a phase proportional to the number of charges of the opposite color linked by a braid (weighted by the orientation of the braid). It has recently been shown that a similar braiding characterizes the solitons for the $(2+1)$-dimensional CP^1 model [60].

Kantor and Susskind [54] argued that a natural candidate for anyons is the dumbbell matter which consists of a pair of charges, g and $\tilde{g}$, one placed on a primary lattice site and the other on a neighboring dual lattice site. They could be viewed as confined pairs of primary and dual lattice particles and could arise in a realistic situation as a tightly bound electron–magnetic flux tube pair, for example. Alternatively, if we view the primary and dual lattices as embedded in an ordinary square lattice, with the primary and dual lattices each being a next-to-nearest neighbor sublattice, we could think of dumbbells as representing confined pairs of ordinary particles. For such a pair the low-energy hopping process would effectively be for one of the members to the pair to hop between sites which are nearest neighbors to the location of the other member of the pair. Thus, effectively, the

[8] Note that, as is characteristic of constrained systems, this state is not normalizable. When taking the norm one should properly view $[\delta(B(x^*))]^2$ as $\delta(B(x^*)) \times$ (volume of gauge group).

lowest energy hopping would transport particles only on one of the next-to-nearest neighbor sublattices.

Here, following Kantor and Susskind, we shall treat dumbbells as elementary extended particles. We denote a primary–dual site pair by $[x, \sigma]$, where x is a primary lattice site, and σ is a vector with entries $\pm\frac{1}{2}$, denoting on which of the four neighboring dual sites the other end of the dumbbell lies. Creation and annihilation operators for dumbbells are denoted by $D^\dagger([x, \sigma])$ and $D([x, \sigma])$. Their commutation relations are given by

$$[D([x, \sigma]), D^\dagger([y, \sigma'])] = \delta_{xy}\delta_{\sigma\sigma'}. \tag{7.24}$$

Note that we have not included the parity image of the dumbbells in this system, because under parity the tilde particles have to undergo charge conjugation. Thus, although the Chern–Simons term preserves parity, the matter Hamiltonian does not, and parity is broken in the dumbbell-gauge field system.

The charge density on a primary site x is given by $\rho(x) = \sum_\sigma D^\dagger D([x, \sigma])$, summed over all neighboring dual lattice sites, where the other end of the dumbbell might reside. We may similarly define a charge density $\tilde{\rho}$ on a dual site by summing $D^\dagger D([x, \sigma])$ over neighboring primary sites. In terms of these variables, we may express the gauge generators (7.18) as

$$\mathscr{G}_\chi = \sum_x \chi(x)\left(g\rho(x) - \frac{1}{\theta}\tilde{B}(x)\right), \qquad \tilde{\mathscr{G}}_{\tilde{\chi}} = \sum \tilde{\chi}(x^*)\left(\tilde{g}\rho(x^*) - \frac{1}{\theta}B(x^*)\right). \tag{7.25}$$

Gauge invariant transport of a dumbbell involves attaching a Wilson line to each component of the dumbbell. The Wilson lines follow parallel paths and can be thought of as a ribbon. The Wilson line operators can be combined to form a "Wilson ribbon" operator

$$W[C, \tilde{C}] = \exp\left(ig \sum_{\text{links of } C} A + i \sum_{\text{links of } \tilde{C}} \tilde{A}\right), \tag{7.26}$$

where C and $\tilde{C}$ are adjacent curves. A gauge invariant dumbbell is given by

$$\phi_{C\tilde{C}}([x, \sigma]) = W[C, \tilde{C}]\, D([x, \sigma]), \tag{7.27}$$

where C and $\tilde{C}$ go from x and $x + \sigma$ to a pair of adjacent base points chosen at one of the corners of the (finite) lattice. Note that this is the automatic and explicit framing of anyonic braids referred to in Section V.

The empty dumbbell vacuum is constructed as in (7.22) with $|0\rangle$ as the state with no dumbbells. From it we construct the state with n dumbbells as

$$\phi_{C_1\tilde{C}_1}([x, \sigma]_1) \cdots \phi_{C_n\tilde{C}_n}([x, \sigma]_n)\, \Psi_0. \tag{7.28}$$

As in the case of the unconstrained matter we first discussed, gauge invariant transport of a dumbbell around a loop $l, \tilde{l}$ has the effect of multiplying the state from the

[9] We arrange the corner of the lattice such that both B and $\tilde{B}$ are on the boundary of the lattice, so that neither B nor $\tilde{B}$ may be encircled by a curve on the other lattice.

left by the Wilson ribbon operator $W[l, \tilde{l}]$. Commuting this through the vacuum, we find the sum of the two phases that were found for the tilde and non-tilde particles, namely, $\exp(ig\tilde{g}\theta$ (number of dumbbells linked by the loop)). It should be noted that dumbbells sitting on top of the ribbon loop give a half linking. Thus monodromy for the dumbbells is exactly that expected for anyons.

Half-monodromy, which gives the statistics phase, may be treated by computing the commutator of two dumbbell–anyon creation operators $\phi_{C_1\tilde{C}_1}([x, \sigma]_1) = W[C_1, \tilde{C}_1]\, D([x, \sigma]_1)$, $\phi_{C_2\tilde{C}_2}([x, \sigma]_2) = W[C_2, \tilde{C}_2]\, D([x, \sigma]_2)$ directly. The general formula for such a commutator is proportional to the sum of the intersection numbers of the two sets of curves, i.e., $\exp ig\tilde{g}\theta(\nu[C_1, \tilde{C}_2] + \nu[\tilde{C}_1, C_2])$. For a pair of Wilson ribbons, starting at a pair of base points such as we have described, this number is simply 1 (giving half of the monodromy calculated above) if the two ribbons do not intersect away from the base point, because there is necessarily a single intersection when the two ribbons separate. For generic additional intersections of the ribbons, the half-monodromy increases by $2g\tilde{g}\theta$, but an exception to this occurs when one of the particles lies on the Wilson ribbon of the other—then, as in the monodromy, only one of the two pairs of Wilson lines in the ribbons intersect, and the half-monodromy phase increases only by $g\tilde{g}\theta$.

Spin for the dumbbells may be treated in a way similar to that of the monodromy, by rotating the dumbbell about one of its ends, using a Wilson loop operator for that end to effect the transport. In this case, the path of the rotating end of the dumbbell links the stationary end, and there is a single intersection of the resulting Wilson loop with one of the Wilson lines in the dumbbell–anyon creation operator. This gives a phase identical to the (generic intersection-less) half-monodromy, $\exp ig\tilde{g}\theta$. Thus the spin-parity of a single dumbbell, i.e., the change in phase of its wavefunction under a 2π-rotation, is identical to the phase the wavefunction acquires under the exchange of adjacent dumbbells. This is the conventional spin–statistics relationship.

The curious way that fractional statistics come out of this model may be understood by examining its continuum limit. This is a model containing two flavors of fermions coupled to two separate gauge fields, which are themselves coupled by a mixed Chern–Simons term. The dumbbell condition may be most straightforwardly achieved by simply setting the two fermion currents equal. To be more specific, let us consider a model of lattice fermions coupled to the Chern–Simons gauge fields with action

$$S = \frac{1}{\theta} S_{\mathrm{CS}} + \int dt \sum_x \left(a^\dagger(x) \left(i\frac{\partial}{\partial t} - gA_0(x) \right) \right.$$
$$\left. \times a(x) + \tilde{a}^\dagger(x^*) \left(i\frac{\partial}{\partial t} - \tilde{g}\tilde{A}_0(x^*) \right) \tilde{a}(x^*) \right) - H_{\mathrm{M}}, \tag{7.29}$$

where the matter field Hamiltonian is given by

$$H_{\mathrm{M}} = \sum_{x,\mu} \Delta_\mu(x)\, a^\dagger(x+\mu)\, e^{igA_\mu(x)} a(x) + \sum_{x^*,\mu} \tilde{\Delta}_\mu(x^*)\, \tilde{a}^\dagger(x^*+\mu)\, e^{i\tilde{g}\tilde{A}_\mu(x^*)} \tilde{a}(x^*) \tag{7.30}$$

with the constraint that the total density of particles on the primary attice is $\frac{1}{2}$ per site and the total density of particles on the dual lattice is also $\frac{1}{2}$ per site. We regard (7.29) and (7.30) as lattice regularization[10] of a relativistic continuum field theory with action

$$S_{\text{cont}} = \int \left(\frac{1}{\theta} \tilde{A} dA + \bar{\psi} i\gamma \cdot \partial \psi + \bar{\tilde{\psi}} i\gamma \cdot \partial \tilde{\psi} + gV \cdot \bar{\psi} \gamma \psi + \tilde{g} \tilde{A} \cdot \bar{\tilde{\psi}} \gamma \tilde{\psi} \right). \tag{7.31}$$

This may be rearranged by writing $A_\mu^{\pm} = A_\mu \pm \tilde{A}_\mu$, and $j_\mu^{\pm} = g j_\mu \pm \tilde{g} \tilde{j}_\mu$, so that the action now becomes

$$S_{\text{cont}} = \int \left(\frac{1}{4\theta} [A^+ dA^+ - A^- dA^-] + \frac{1}{2} A^+ \cdot j^+ \right.$$
$$\left. + \frac{1}{2} A^- \cdot j^- + \bar{\psi} i\gamma \cdot \partial \psi + \bar{\tilde{\psi}} i\gamma \cdot \partial \tilde{\psi} \right) \tag{7.32}$$

and we may complete the square to find

$$S_{\text{cont}} = \int \frac{1}{4\theta} \left[\left(A^+ + \frac{1}{\partial^2} \theta \, dj^+ \right) d \left(A^+ + \frac{1}{\partial^2} \theta \, dj^+ \right) \right.$$
$$\left. - \left(A^- - \frac{1}{\partial^2} \theta \, dj^- \right) d \left(A^- - \frac{1}{\partial^2} \theta \, dj^- \right) \right] \tag{7.33}$$
$$- \frac{\theta}{4} \left[j^+ \cdot \left(\frac{1}{\partial^2} dj^+ \right) + j^- \cdot \left(\frac{1}{\partial^2} dj^- \right) \right] + \bar{\psi} i\gamma \cdot \partial \psi + \bar{\tilde{\psi}} i\gamma \cdot \partial \tilde{\psi}. \tag{7.34}$$

The gauge fields may now be integrated out, and the terms that are left over may be recognized as the expression for the linking number of two trajectories, written in terms of their currents [10]. We now impose the dumbbell constraint, which amounts to $j^- = 0$ and confines charges and tilde-charges into bound pairs. Given two such bound states, we see that, if their classical paths intertwine, this action has the effect of assigning an extra anyonic phase to the amplitude. Thus, the effect of the dumbbell constraint in this continuum limit of dumbbell Chern–Simons theory is to both confine the charges and to select out only one combination of the gauge fields as influencing the dynamics, recovering ordinary continuum Chern–Simons theory in a unique way.

VIII. Discussion

We have presented an exact map of lattice Chern–Simons theory coupled to fermions onto anyons interacting via a Z_N gauge field. This establishes that exact

[10] Here again we have inserted classical phases into the hopping amplitude in the Hamiltonian. We remind the reader that if we choose these classical phases to that $\prod \Delta$ for each plaquette is (-1) the spectrum of each part of the Hamiltonian is that of two species of two-component relativistic fermions.

anyonization is possible, without the singularities and other problems that earlier plagued continuum formulations and, further, without the strong interactions that obscured the meaning of previous results. Although the lattice representation of the Chern–Simons term in our formalism appears to be non-local, in general (with a local continuum limit), we showed that it may always be chosen to be local and we presented a formula for the local, gauge invariant lattice representation of the Chern–Simons term which permits exact anyonization. This allows us to resolve the question of the origin of the statistics phase—it is shown that this phase may be regarded as coming from the half-intersection of the Wilson lines at their common base point. This formulation further makes it possible to regard the theory as a geometric theory, of a line bundle over the lattice. In addition, we have constructed anyonic field operators in an exotic and interesting version of lattice Chern–Simons theory, which sits on a lattice very naturally suited to Chern–Simons theory, and in which the framing of the braids is made explicit.

Further, we have presented a rigorous proof of off-diagonal long-range order for a non-Cooper-pairing fermionic system, Chern–Simons electrodynamics on a lattice with bosonic statistics parameter. This connection with a well-understood system has allowed us to find a new numerical signal for superfluidity, in terms of a bound on the anyon correlation function as a function of q. It is particularly interesting numerically in that it relates the large volume effects (which are the ones of interest) to effects occurring at finite volume, which are more easily calculable numerically.

It is yet possible that the proof of superfluidity can be extended beyond the case where the statistics parameter α is an odd integer, by obtaining the appropriate bound on the fermion correlator. The advantages of the lattice Chern–Simons theory formulation are simply that a direct comparison of the two correlators is possible, for example, in the path integral representation, where differences in the Hilbert space do not appear and the two models of interest differ only by the value of the statistics parameter and filling fraction. Further, the path integral, by discretizing time, can be easily made well defined. The difficulty with such an approach is that the phenomenon of superfluidity is not expected to be generic to anyons, but rather should occur only at specific values of the statistics parameter and filling fraction. Thus the correlator, as a function of these parameters need not be smooth at all. In fact it probably is not, which rules out, for example, a rigorous perturbation analysis.

Given the geometrical nature of the lattice Chern–Simons term which we have discussed, it would be interesting to see whether features of the continuum theory such as duality [60, 61] of certain Bose–Chern–Simons systems or the perturbative arguments for anyon superconductivity [23–29] are operative on the lattice.

APPENDIX: CALCULATION OF $\theta_{C_x}(x, y) - \theta_{C_y}(y, x)$

Consider two points $x, y \in \mathscr{L}$ and two lattice curves C_x and C_y, lengths N_x and N_y, beginning at the base point B, and with endpoints x and y, respectively:

$C_x(0) = C_y(0) = B$, $C_x(N_x) = x$. $C_y(N_y) = y$. We restrict our attention to curves C_x and C_y which do not wind around or intersect their respective vertex points, i.e., C_x does not wind around or intersect y, and C_y does not wind around or intersect x. Also, we assume that the vertical coordinates x_2 and y_2 satisfy $y_2 - x_2 > 0$. The case of nonzero winding number and arbitrary relative positions for x and y may be treated by a simple addition to the following argument.

$\theta_C(x, y)$ is defined in terms of a contour sum over a function f_i, itself defined, on lattice links, as $f_i = \hat{d}_i^{\perp} g$, where g is minus the Green's function of the lattice Laplacian $d \cdot \hat{d}$: $d \cdot \hat{d} g(x) = -\delta(x)$. f_i is the field profile of a vortex, satisfying $\hat{d}_i f_i(x) = 0$, $-d_i^{\perp} f_i(x) = \delta(x)$, centered at $x = 0$. In terms of f_i, the definition of the function $\theta_C(x, y)$ is

$$\theta_C(x, y) = 2\pi \sum_{l \in C}^{x} dl_i f_i(l - y), \tag{A.1}$$

i.e., a contour sum over C, starting at B and ending at x, of the vortex function f_i, whose center has been shifted to y. Note that this is directly analogous to the definition of g, f_i, and θ in the continuum. However, the reflection properties of f_i are altered slightly. Instead of $f_i(x) = -f_i(-x)$, as in the continuum, we find

$$f_i(x) = -S_i^{\perp} f_i(-x) \tag{A.2}$$

(with no sum on i).

The curl property of f_i allows us to freely deform our contour, as long as we leave the endpoints fixed and do not change the winding number around the center of the vortex, which sits at y.[11] Therefore, deform the contour C_x in the following way (assume B sits at radius R on the positive $\hat{1}$ axis, and at the end of the calculation we take $R \to \infty$): The curve C_x rises from B a vertical distance y_2, where y_2 is the component of y in the $\hat{2}$ direction. The contour then travels in the $-\hat{1}$ direction, directly toward the center of the vortex, at y, until it reaches the point $y + \hat{1}$. Finally, the contour travels down to x in an almost arbitrary way, arbitrary save that its penultimate point is $x + \hat{1}$. We assign notation to the separate parts of C_x: the rising portion of length y_2 we call R_x, the middle, horizontal portion we call L_x, and the final, almost arbitrary piece we call F_x. We therefore have $C_x = R_x \cup L_x \cup F_x$. The result is shown in Fig. A-1.

We deform C_y in a similar way. It begins, however, with an additional horizontal piece, moving a distance $y_1 - x_1$ in the $-\hat{1}$ direction (or $x_1 - y_1$ in the $+\hat{1}$ direction). It then, in direct analogy with C_x, rises a vertical distance x_2 and moves a horizontal distance $B - y_1$ in the $-\hat{1}$ direction, ending at the point $x + \hat{1}$. These

[11] As in the continuum, we also must not allow the contour to go through the center of the vortex. Actually, the following argument must change slightly to handle the cases in which $y = x + \hat{2}$ or $y = x + 2\hat{2}$, where we will not be able to avoid hitting y with the contour that this construction gives. The cost of this error, however, is calculable and gives an extra 2π which is not there for proper contours. Thus the formula (A.5) which we will eventually derive will be valid generally.

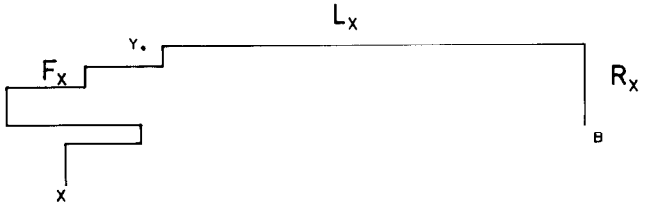

FIG. A-1. The summation contour which defines $\theta_{C_x}(x, y)$.

pieces we name I_y, R_y, and L_y, respectively. These pieces have been chosen so that contour summation over these parts of the curve may either be cancelled pairwise with terms from C_x, or neglected because they are finite in length and sit infinitely far from the center of the vortex f_i. We would like to do something similar with the last part of C_y, i.e., use the skewed reflection property of f_i (A.2). To do so, we attempt to use a reflected version of F_x, denoted F_x^*, to connect the endpoint of L_y with x. We must reflect so that individual terms of the contour sum cancel exactly, i.e., taking account of the extra shift in the reflection property for f_i. When we do so, we find that F_x^* does not begin at the endpoint of L_y, nor does it end at x. Rather, its starting and ending points are both shifted from these points by the vectors $\hat{1}+\hat{2}$ and $\hat{1}-\hat{2}$, respectively. It is therefore necessary to include two additional pieces, D_1 and D_2, that remedy this by connecting the endpoint of L_y to the startpoint of F_x^* and the endpoint of F_x^* to x. Thus we have $C_y = I_y \cup R_y \cup L_y \cup D_1 \cup F_x^* \cup D_2$. The result is shown in Fig. A-2.

Using these contours, we now may directly evaluate the two contour sums in $\theta_{C_x}(x, y) - \theta_{C_y}(y, x)$. They have been arranged so that almost everything cancels, or may be neglected as $R \to \infty$. Clearly, the sums over the pieces I_y, R_x, and R_y are negligible, because $f_i(x)$ is of order $1/R$ for every one of the very finite number of terms in these contour sums. We have already seen that the contours L_x and L_y have been chosen so that their contributions precisely cancel, as have the contours F_x and F_x^*. What remains are the two terms D_1 and D_2. The first is just $-\pi$, precisely the continuum answer. This is true because, on reflecting D_1, using the skewed reflection property (A.2), we actually arrive at precisely the closed contour sum defined by $-d_i^{\perp} f_i(0)$, which is -2π. The sum over D_2, however, is a

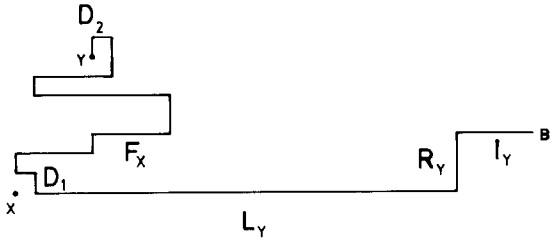

FIG. A-2. The summation contour which defines $\theta_{C_y}(y, x)$.

lattice artifact and is equal to $-f_2(y-x+\hat{1})-f_1(y-x)$ or, when $x \neq y$, $-(\frac{1}{2})[f_2(y-x+\hat{1})+f_1(y-x)+f_1(y-x+\hat{2})+f_2(y-x)]$. This function, which from the reflection property of f_i is an odd function of $y-x$, is denoted by $\xi(y-x)$. Thus we have

$$\theta_{C_x}(x, y)-\theta_{C_y}(y, x)=-\pi+\xi(x-y). \tag{A.3}$$

We now consider the case of more general curves, i.e., with a more general winding number around the respective centers of the vortices and with arbitrary positions for the points x and y. In general, windings add or subtract 2π from the answer found above,

$$\theta_{C_x}(x, y)-\theta_{C_y}(y, x)=-\pi+\xi(x-y)+2\pi(\omega(C_x, y)-\omega(C_y, x)), \tag{A.4}$$

where $\omega(C, p)$ denotes the winding number of a curve C around a point p. But, in addition, the right-hand side of (A.4) is further modified when the sign of the relative height $\operatorname{sgn}(x_2-y_2)$ is reversed. The construction still gives a calculable answer; in fact, it gives the same answer, but one of the curves constructed will contain an extra, unintended winding, which has to be subtracted off to obtain the analog of (A.3) for this case. Thus, when x and y change places, (A.3) becomes

$$\theta_{C_x}(x, y)-\theta_{C_y}(y, x)=\pi+\xi(x-y) \tag{A.3'}$$

and, in general,

$$\theta_{C_x}(x, y)-\theta_{C_y}(y, x)=\operatorname{sgn}(x_2-y_2)\pi+\xi(x-y)+2\pi(\omega(C_x, y)-\omega(C_y, x)). \tag{A.5}$$

We note here that the function ξ has appeared previously in the context of lattice angle deficits, in work by Luscher [32].

We wish to re-express this in terms of intersections between the curves themselves. To do so, we note that the presence of the base point makes this difficult, inasmuch as the intersection number of the two curves may be changed without affecting the windings of the curves, as long as a compensating change is made in the way that the curves leave the basepoint. In the following, we shall redefine the contour sum in such a way that this extra freedom is nailed down.

We impose the additional condition on all contours that they leave the basepoint vertically, rise until they are at the same vertical coordinate as the center of the vortex (as in R_x or R_y), and afterwards run in towards the vortex. The curves may do anything after that. This amounts to demanding that the curves leave from a base line at infinity on the $\hat{1}$ axis, at a vertical coordinate equal to that of their endpoints. Note that the curves C_x and C_y do not satisfy this condition, and therefore $\theta_{C_x}(x, y)-\theta_{C_y}(y, x)$ as calculated above will differ by 2π from the formula that we are about to derive.

With curves obeying the above condition, the case treated in the first section of this appendix corresponds to no intersections of the curves at all. We may create intersections by pushing the upper curve down so that it crosses the lower curve, and if we do not try to create windings, this curve will in fact cross a second time,

on its way back. These two intersections actually have an opposite handedness, and from this example we see that it is really the intersection number weighted by this handedness, or the signed intersection number, that we are interested in. The handedness of an intersection between two directed curves C and C' is simply the sign of the cross product of their tangent vectors at the crossing. If the cross product points out of the page, then the sign of the intersection is positive, and if it points into the page, then the sign of the intersection is negative. It is this signed intersection number that keeps track of the difference between the winding numbers of C_x around y and C_y around x, i.e., for curves restricted as above, $2\pi(\omega(C_x, y) - \omega(C_y, x)) = 2\pi\nu(C_x, C_y)$. Thus, we may rewrite the formula (A.5) in terms of the signed intersection number as

$$\theta_{C_x}(x, y) - \theta_{C_y}(y, x) = \operatorname{sgn}(x_2 - y_2)\pi + \xi(x - y) + 2\pi\nu(C_x, C_y). \tag{A.6}$$

This is the formula that we will find useful in the text.

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