CHIRAL DYNAMICS AND FERMION MASS GENERATION IN THREE DIMENSIONAL GAUGE THEORY

M. C. Diamantini[1] and P. Sodano[1]

Dipartimento di Fisica and Sezione I.N.F.N., Universitá di Perugia
Via A. Pascoli, 06100 Perugia, Italy

G. W. Semenoff[2]

Department of Physics, University of British Columbia
Vancouver, B.C., Canada V6T 1Z1

Abstract

We examine the possibility of fermion mass generation in 2+1–dimensional gauge theory from the current algebra point of view. In our approach the critical behavior is governed by the fluctuations of pions which are the Goldstone bosons for chiral symmetry breaking. Our analysis supports the existence of an upper critical N_F and exhibits the explicit form of the gap equation as well as the form of the critical exponent for the inverse correlation length of the order parameter.

The possibility of dynamical generation of fermion masses is of fundamental importance to our present view of quantum chromodynamics (QCD) as a theory of the strong interactions. Central to our understanding of this phenomenon is the existence of a critical coupling. When fermions have a sufficiently strong, attractive interaction there is a pairing instability and the resulting condensate breaks some of the flavor symmetries, generates quark masses and represents chiral symmetry in the Nambu-Goldstone mode. This idea dates back to the earliest models for chiral symmetry breaking [1] and is prevalent in the modern literature [2].

Recently the issue of critical coupling has been investigated in 2+1 dimensional gauge theories [3, 4, 5]. These theories provide toy models which exhibit simpler behavior than their 3+1–dimensional relatives [6, 7] and are also of interest as effective field theories for some condensed matter systems [8]. Typically, their dimensionless expansion parameter is $1/N_F$, where N_F is the number of quark flavors [3]. Using Schwinger-Dyson equations in the $1/N_F$ approximation for QED and QCD, [3] have found that there is a critical $N_F^{\rm crit}$ such that when $N_F < N_F^{\rm crit}$ chiral symmetry is broken and when $N_F > N_F^{\rm crit}$ it is not broken and quarks remain massless.

In the case of QED, this result has been the subject of some debate [4, 5, 9, 10]. Either an improved ladder approximation [5, 10] or renormalization group computation [4] find no critical behavior and that chiral symmetry is broken for arbitrarily large N_F. There are, however, numerical simulations [11] of 2+1– dimensional QED which find an $N_F^{\rm crit}$ remarkably close to that obtained by [3].

In this Letter we shall present further support for the existence of $N_F^{\rm crit}$. We shall advocate a picture which is complementary to that of critical attractive quark-quark interactions and in which the critical behavior of the chiral symmetry breaking phase transition is governed by the fluctuations of the pions which are the Goldstone bosons for broken continuous flavor symmetries. We shall argue that an upper critical N_F is natural since the order parameter is renormalized by the $N_F^2/2$ pions with classical coupling constant $\sim 1/N_F N_C$ where N_C is the number of quark colors. Their fluctuations are strong enough to destroy the ordered state when $N_F = N_F^{\rm crit} \sim N_C$.

We are partially motivated by our recent study of strong coupling gauge theory [12] on the lattice. We showed that, in the strong coupling limit, a Hamiltonian lattice gauge theory with N_C colors and $N_F/2$ lattice flavors of staggered fermions (because of fermion doubling this corresponds to N_F

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continuum flavors of 2–component spinors in 2+1–dimensions) is effectively a $U(N_F/2)$ quantum antiferromagnet, with representations determined by N_C and N_F. We also identified chiral symmetry breaking with the formation of either commensurate $U(1)$ charge density waves or $SU(N_F/2)$ spin density waves, i.e. Néel order. We found that the cases where $N_F/2$ is an odd or even integer are quite different.

When $N_F/2$ is odd the strong coupling limit necessarily breaks chiral symmetry, no matter how large N_F is and the condensate is a $U(1)$ charge density wave. The ground state has a staggered structure with the $SU(N_F/2)$ representations given by the Young tableau with N_C columns and $N_F/4+1/2$ rows (and U(1) charge 1/2) on even sites and N_C columns and $N_F/4 - 1/2$ rows (and U(1) charge -1/2) on odd sites. (This was strictly true for $U(1)$ gauge theory, and likely for $U(N_C)$ and $SU(N_C)$ gauge theory where it could be proved only with some additional assumptions about translation invariance.)

We found that when $N_F/2$ is an even integer, there is no $U(1)$ charge density wave and the representation at each site was given by the Young tableaux with N_C columns and $N_F/4$ rows.

In either case of even or odd $N_F/2$, there could be Néel order, which also breaks chiral symmetry. Quantum antiferromagnets with the kinds of representations we considered have been analyzed in [13] where they found that, for small enough N_F, the ground state is ordered. Also, in the generic case, when N_F is increased there is a phase transition with $N_F^{\text{crit}} \sim N_C$ to a disordered state. In this picture, the large N_C limit is the classical limit where the Néel ground state is favorable and the small N_C and large N_F limit is where fluctuations are large and disordered ground states are favored.

For example, the $SU(2)$ antiferromagnet with spin j corresponds to 4-flavor QCD with color group U(2j) and in particular to QED when j=1/2. It has a Néel ordered ground state for any j, corresponding to chiral symmetry breaking in the strong coupling limit of QCD. However, an $SU(N_F/2)$ antiferromagnet with N_F a large multiple of 4 and in a representation of $SU(N_F/2)$ given by a Young tableau with a single column of $N_F/4$ boxes corresponds to strong coupling QED with N_F flavors of fermions. It is known that the ground state of this system is disordered with several competing flux and dimer phases [8, 13]. For some intermediate N_F between 2 and ∞ the antiferromagnet has a phase transition where Néel order is lost.

It is tempting to conclude that this critical behavior of antiferromagnets

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is related to the critical behavior of continuum gauge theory found in [3] using Schwinger-Dyson equations. In the following we wish to examine this question further in the continuum by analyzing the dynamics of the effective field theory for pions in the phase with broken chiral symmetry.

Our lattice results seem to imply that there is a subtle difference between the cases where N_F is an even or odd multiple of 2. It is not clear whether this is a fundamental difference in the continuum theory too, or merely an artifact of forcing gauge theory to live on a lattice. The numerical lattice simulations of QED in [11] use staggered Euclidean fermions and can therefore only study the case when N_F is a multiple of 4. For compact continuum QED we argued in [12] that, if one uses Polyakov's idea [18] of taking the Georgi-Glashow model with spontaneous symmetry breaking $SO(3) \to U(1)$ to obtain a photon with compact $U(1)$ gauge group, then it is known [15] that the minimal number of 2-component spinors the photons can couple to, consistent with gauge and parity invariance, is 4. Thus, if continuum QED is to be compact, N_F is a multiple of 4. This could also apply to $U(N_C)$ gauge theory because of the $U(1)$ subgroup. However, the only such restriction on QCD with $SU(N_C)$ gauge group constrains N_F to be even.

We shall consider QCD with gauge group $SU(N_C)$ and N_F ($=2\times$integer) flavors of massless 2-component quarks in Euclidean space

$$
S = \int d^3x \left(\frac{1}{4e^2\Lambda} \sum_{a=1}^{N_C^2-1} F_{\mu\nu}^a F_{\mu\nu}^a + \sum_{\alpha=1}^{N_F} \bar{\psi}^\alpha \gamma^\mu (i\partial_\mu + A_\mu)\psi^\alpha \right) \tag{1}
$$

where e^2 is the dimensionless coupling constant and Λ is the ultraviolet cutoff. This action has $U(N_F)$ global flavor symmetry and also a Z_2 parity symmetry under the replacement $(A_1, A_2, A_3)(x) \to (-A_1, A_2, A_3)(x')$, $\psi(x) \to \gamma_1\psi(x')$, $\bar{\psi}(x) \to -\bar{\psi}(x')\gamma_1$ with $x' = (-x_1, x_2, x_2)$. We shall assume that the ultraviolet regularization preserves parity.

An order parameter for the $U(N_F) \times Z_2$ symmetry breaking is the quark bilinear

$$
M^{\alpha\beta}(x) = \bar{\psi}^\alpha(x)\psi^\beta(x) \tag{2}
$$

M is a Hermitean matrix and transforms under $U(N_F)$ as $M \to gMg^\dagger$ and under parity as $M \to -M$. (This is in contrast with its counterpart in 3+1–dimensions, $\mu = \bar{\psi}_L\psi_R$ which is a complex matrix and transforms under

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$SU_R(N_F) \times SU_L(N_F)$ as $\mu \to g\mu h^\dagger$.) Flavor symmetry breaking is governed by the effective Landau-Ginsburg action

$$S_{eff} = \int d^3x \ \text{tr} \left(c_1 \partial_\mu M \partial_\mu M + c_2 M^2 + c_3 M^4 + \ldots \right) \qquad (3)$$

Note that, in the large N_C limit, the coefficients c_i in (3) are fermion loops with meson operator insertions which are naturally of order N_C [16].

We shall consder the symmetry breaking pattern $U(N_F) \times Z_2 \to U(n) \times U(N_F - n)$. In this case, M has a constant vacuum expectation value

$$M_0 = <M> = \ \text{const diag} \ (1, 1, 1, \ldots, -1, -1, -1) \qquad (4)$$

where there are n 1's and N_F-n -1's. We shall show that the case where $n = N_F/2$, for which a residual parity symmetry can be defined, is dynamically favorable.

We are interested in the dynamics of Goldstone bosons which are described by a sigma model with target space the Grassmannian

$$\frac{U(N_F) \times Z_2}{U(n) \times U(N_F - n)} \quad .$$

With the ansätz $M(x) = g(x)M_0 g^\dagger(x)$ we obtain the sigma model

$$S_{\text{eff}} = \int d^3x \frac{\Lambda N_C}{2f^2} \ \text{tr} \ \left([g\partial_\mu g^\dagger, M_0][g\partial_\mu g^\dagger, M_0] \right) + S_{WZ} \qquad (5)$$

where we have renamed the coefficient, which here plays the role of coupling constant and we have extracted its natural order in N_C. S_{WZ} is a Wess-Zumino term which must be added to the sigma model action in order to break an unwanted discrete symmetry and to obtain a sensible current algebra [17]. (5) is equivalent to the gauged principal chiral model

$$S_{\text{eff}} = \int d^3x \frac{\Lambda N_C}{f^2} \ \text{tr} \ \left((Dg)^\dagger \cdot (Dg) + i\lambda(g^\dagger g - 1) \right) + S_{CS}[V,W] \qquad (6)$$

where $D = \partial + i(V + W)$ and V and W are Hermitian gauge fields which have components in the upper left $n \times n$ block and in the lower right $(N_F - n) \times (N_F - n)$ block respectively. In (6) the Chern-Simons action is

$$S_{\text{CS}}[V,W] = i\frac{\theta}{4\pi} \int d^3x \text{tr} \left(VdV - WdW + \frac{2}{3}V^3 - \frac{2}{3}W^3 \right) \qquad (7)$$

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and we have introduced an $N_F \times N_F$ Hermitean Lagrange multiplier field λ to enforce the constraint $gg^\dagger = 1$. Eliminating V and W in (6) using their equations of motion yields (5), up to higher derivative terms which come from approximating the Wess-Zumino term by the Chern-Simons action. In [17] it was argued that, the model described by (6) has features remarkably similar to the commonly accepted features of the Skyrme model of 3+1-dimensional QCD, such as solitons which have Fermi statistics and behave like baryons. They also argued that to obtain the correct current-current commutation relation, one must set $\theta = N_C$. To be general, we shall keep θ arbitrary in the following analysis.

In order to study the quantum properties of this model, we first note that the field g appears quadratically and can be integrated to get the effective theory

$$S_{\text{eff}} = N_F \; \text{TR} \; \ln(-D^2 + i\lambda) - \int d^3x \frac{i\Lambda N_C}{f^2} \; \text{tr} \; \lambda + S_{CS}[V, W] + S_{\text{ghost}} \quad (8)$$

where, to fix the gauge freedom and properly define the quantum problem we have added the Faddeev-Popov ghost action,

$$S_{\text{ghost}} = \int d^3x \; \text{tr} \left(\frac{1}{2\alpha}(\partial V)^2 + \frac{1}{2\beta}(\partial W)^2 + \partial c^\dagger (\partial + iV)c + \partial d^\dagger (\partial + iW)d \right)$$

As in the standard approach to sigma models [18, 19], the remaining integral over λ, U and V is done by saddle-point approximation. It is assumed that the gauge fields are zero at the saddle point. The saddle point value of λ provides a mass for the chiral field g in (6). When this mass is non-zero g fluctuates about $g=0$ and the model is disordered. When the saddle point is at zero, we obtain the ordered phase. Assuming a constant saddle point and putting $i\lambda = \mu^2$, we obtain the gap equation

$$\frac{N_F N_C}{f^2} \Lambda = \frac{N_F^2 \Lambda}{2\pi^2} \left(1 - \frac{\mu}{\Lambda} \arctan \frac{\Lambda}{\mu} \right) \quad (9)$$

A solution of this equation exists and the model is disordered if

$$N_F \geq 2\pi^2 \frac{N_C}{f^2} \quad (10)$$

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with the equality giving the condition for criticality. For fixed f^2 and N_C we can interpret this as an equation for critical N_F. When N_F exceeds $N_F^{\text{crit}} = 2\pi^2 N_C/f^2$, the chiral symmetry breaking condensate is unstable.

Note that, unlike the case of the more conventional O(N) non–linear sigma model, the saddle point approximation in the present case is not controlled by any small parameter such as $1/N$. This is because, even though the effective coupling constant of the saddle-point method is $\sim 1/N_F^2$, there remain of $\sim N_F^2$ degrees of freedom. The correct large N_F expansion of (6) would involve the topological expansion where, like the large N_C expansion of QCD [16], it is necessary to sum all planar graphs to obtain the leading order.

In spite of this, we have two reasons to believe in the validity of (10). First of all, as we shall see in the following, corrections from the next order in the saddle point approximation are indeed smaller than the leading order. Secondly, alternative to the saddle point analysis, we could do a weak coupling expansion. If N_F is large enough, criticality in (10) is obtained in the weak coupling region.

The computation of the gap equation to next order is somewhat sophisticated. We must expand (8) to quadratic order in V, W, λ, drop linear terms [20] and perform the functional integral in the Gaussian approximation.

$$
S_{\text{eff}} = N_F^2 V \int^\Lambda \frac{d^3k}{(2\pi)^3} \ln(k^2 + \mu^2) - \frac{\Lambda N_C N_F \mu^2}{f^2} V + N_F^2 V \int^\Lambda \frac{d^3k}{(2\pi)^3} \frac{1}{k^2 + \mu^2} +
$$
$$
+ N_F \int \text{tr} \left(\frac{1}{2} \lambda \Delta \lambda + \frac{1}{4} F^V \Pi F^V + \frac{1}{4} F^W \Pi F^W + \dots \right) + S_{CS} + S_{\text{ghost}} \quad (11)
$$

where

$$
\Delta(k, \mu) = \frac{1}{4\pi|k|} \arctan \frac{|k|}{2|\mu|} \quad ,
$$

$$
\Pi(k, \mu) = \frac{1}{8\pi} \left(\frac{2|\mu|}{k^2} + \frac{k^2 + 4\mu^2}{|k|^3} \arctan \left(\frac{|k|}{2|\mu|} \right) \right) \quad ,
$$

V is the volume, we have set $\lambda \to -i\mu^2 - i\delta\mu^2 + \lambda$ and $\delta\mu^2 = -2\Lambda^2/\pi^2$ is a counterterm which arises from an additive shift of λ and which is necessary to cancel a quadratic divergence. This infinite shift of the Lagrange multiplier field is a standard feature of sigma model renormalization [19]. The result of integrating the Gaussian fluctuations is

$$
S_{\text{eff}}/V = -\frac{\mu^2 N_F N_C}{f^2} \Lambda + \int^\Lambda \frac{d^3k}{(2\pi)^3} \left(N_F^2 \ln(k^2 + \mu^2) + N_F^2 \delta\mu^2 \frac{1}{(k^2 + \mu^2)} + \right.
$$

$$+ \frac{N_F^2}{2} \ln \Delta + \frac{n^2 + (N_F - n)^2}{2} \ln \left(k^2 \Pi^2 + (\theta/4\pi)^2 \right) + \ldots \bigg)$$

$$= \left(-\frac{N_C}{N_F f^2} + \frac{1}{2\pi^2} - \frac{1}{\pi^4} + \frac{2}{\pi^2} \frac{1 + 2n^2/N_F^2 - 2n/N_F}{1 + (8\theta/\pi^2)^2} \right) N_F^2 \Lambda \mu^2 -$$

$$-\frac{\pi}{3} \left(1 - \frac{2\pi^2 - 8}{\pi^4} \ln \frac{\Lambda}{\mu\alpha} + \frac{16}{\pi^4} \frac{1 + 2n^2/N_F^2 - 2n/N_F}{1 + (8\theta/\pi)^2} \ln \frac{\Lambda}{\mu\beta} \right) N_F^2 |\mu|^3 + \ldots \, (12)$$

Here, α and β are (unknown) constants. The method for computing the integrals in described in the Appendix. If interpreted as an effective potential for μ, S_{eff} is upside–down and apparently unstable. This originates with subtleties in dealing with complex saddle points. We refer the reader to the standard literature on the subject [21]. The corrected formula for the critical line is

$$N_F^{\text{crit}} = 2\pi^2 \frac{N_C}{f^2} \left(1 - \frac{2}{\pi^2} + \frac{4(1 + 2n^2/N_F^2 - 2n/N_F)}{1 + (8\theta/\pi)^2} \right)^{-1}$$

When θ (which is equal to N_C) is large, the right hand side differs by about 20 percent from the leading order estimate. The largest N_F^{crit} occurs when $n = N_F/2$. Thus, the first and therefore most stable ordering which occurs as as we lower N_F is the parity symmetric phase $n - N_F/2$.

We can also interpret the logarithms in (12) as changing the exponent, $|\mu|^3 \to |\mu|^{3(1+\gamma)}$, where

$$\gamma = \frac{8}{3\pi^4} \left(1 - \frac{\pi^2}{4} + \frac{1 + 2n^2/N_F^2 - 2n/N_F}{1 + (8\theta/\pi)^2} \right)$$

and the gap equation has solution

$$\mu \sim \left(N_F - N_F^{\text{crit}} \right)^{1/(2+3\gamma)}$$

which exhibits the critical exponent (again a small correction of the leading order result $1/2$) for the inverse correlation length of the order parameter.

In conclusion, we note that the beta function of the Grassmannian sigma model has been computed using the ϵ–expansion about 2 spacetime dimensions in [22]. In their notation it is

$$\beta(t) = (d - 2)t - N_F t^2 - \ldots$$

and has an ultraviolet stable fixed point at $t = (d-2)/N_F$. In our notation the coupling constant is $t = f^2/2\pi^2 N_C$, and in the leading order, the critical point (10) occurs at the zero of the beta–function. This is an ultraviolet stable fixed point (it cannot be infrared stable since the flow to low momentum should be interrupted by mass generation).

Appendix: Calculation of Integrals

As an example of the integrations needed to find the quantum corrections to the gap equation, consider the integral $I[\Lambda/\mu] = \int_0^{\Lambda/2\mu} dx\ x^2 \ln \frac{2}{\pi} \arctan x$ which we want to evaluate in the limit $\Lambda/\mu \to \infty$. To evaluate the integral, we first extract the divergent parts as

$$I[\Lambda/\mu] = \int_0^{\Lambda/2\mu} dx\ x^2 \left(\ln \frac{2}{\pi} \arctan x + \frac{2}{\pi x} + \frac{1}{\pi^2 x^2} - \frac{2\pi^2 - 8}{3\pi^3 x^2 (x+\alpha)} \right) +$$

$$- \frac{\Lambda^2}{4\pi\mu^2} - \frac{\Lambda}{\pi^2 \mu} + \frac{2\pi^2 - 8}{3\pi^3} \ln(\Lambda/2\mu\alpha)$$

In the first term the infinite cutoff limit can safely be taken, to get a function of α. However, α is an arbitrary number, so the integral can be parameterized as

$$I[\Lambda/\mu] = -\frac{\Lambda^2}{4\pi\mu^2} - \frac{2\Lambda}{\pi^2 \mu} + \frac{2\pi^2 - 8}{3\pi^3} \ln(\Lambda/2\mu\tilde{\alpha})$$

where $\tilde{\alpha}$ is a fixed, but unknown (and irrelevant) constant.

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