

# Topological Filters for Solitons in Coupled Waveguides Networks

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## Abstract

We study the propagation of discrete solitons on chains of coupled optical waveguides where finite networks of waveguides are inserted at some points. By properly selecting the topology of these networks, it is possible to control the transmission of traveling solitons: we show here that inhomogeneous waveguide networks may be used as filters for soliton propagation. Our results provide a first step in the understanding of the interplay/competition between topology and nonlinearity for soliton dynamics in optical fibers.

The discovery of solitons in optical fibers, three decades ago [1], stimulated a huge amount of work aimed at using solitons for high speed communications [2]. Many experiments evidenced the role of the Kerr nonlinearity in allowing for the propagation over long-distances of solitons in optical fibers [2,3]; however, this nonlinear paradigm has not yet demonstrated decisively its advantages over other more conventional signal propagation schemes. If one introduces a spatial inhomogeneity in the field equations by the linear compression mechanism, one could stabilize the soliton propagation [4]. Motivated by this, one is lead to investigate the so-called *dispersion-managed* nonlinear Schrödinger equation [5]

$$i \frac{\partial E}{\partial z} = -\frac{\beta(z)}{2} \frac{\partial^2 E}{\partial t^2} - \nu |E|^2 E, \quad (1)$$

where  $E(z, t)$  is the electric field,  $\nu$  the Kerr nonlinearity,  $z$  the propagation direction and  $t$  the retarded time.  $\beta(z)$  is the term responsible for dispersion management. In Eq.(1) the term  $\beta(z)$  plays the role of a spatial modulation of the kinetic term; upon discretization, this naturally leads to consider a network arranged in a non-translational invariant topology.

In this paper we shall consider a discrete version of the nonlinear Schrödinger equation (1), describing an array of coupled optical waveguides [6]. Namely, we shall show that, by choosing a pertinent network of coupled waveguides, one may engineer soliton filters for discrete soliton propagation. We focus on waveguide networks, built by adding a finite discrete network of optical waveguides (the graph  $G^0$ ) to a chain (see Fig.1). By means of a general criterion derived within a linear approximation for a relevant class of solitons [7], denoted as *large-fast solitons*, we show the possibility of controlling the soliton scattering, by suitably choosing the topology of  $G^0$ . The analytical findings are in agreement with the numerical study of the full nonlinear evolution equation.

The discrete nonlinear Schrödinger equation (DNLSE), for a general network of coupled optical waveguides, reads

$$i\frac{\partial E_n}{\partial z} = -\sum_j \beta_{n,j} E_j + \Lambda |E_n|^2 E_n. \quad (2)$$

Here  $E_n(z)$  is the electric field in the  $n$ th waveguide and  $\Lambda$  is proportional to the Kerr nonlinearity. The normalization is chosen to be  $\sum_n |E_n(z)|^2 = 1$ . In Eq.(2),  $\beta_{n,j}$  is proportional to the mode overlap of the electric fields of the waveguides  $n$  and  $j$  [6,9] and it is non-0 only if  $n$  and  $j$  are neighbours waveguides. If identical waveguides are arranged to form a chain, then Eq.(2) assumes the usual form  $i\partial E_n/\partial z = -\beta_c(E_{n+1}+E_{n-1})+\Lambda |E_n|^2 E_n$  where  $\beta_{n,n\pm 1} = \beta_c$ . If, on the other hand, the waveguides are arranged on the sites of a non-translational invariant network, a space modulation of the kinetic term occurs, even if  $\beta$  does not depend on  $z$ . As an example, one can consider the waveguide geometry depicted in Fig.1. Of course, one can imagine and engineer a huge variety of network topologies; our aim here is to show that the network topology may be used to engineer a novel class of filters for soliton motion over spatially inhomogeneous networks of coupled waveguides.

It is well known that the DNLS on a homogeneous chain is not integrable [6]; nevertheless, soliton-like wavepackets can propagate for a long time [10], and the stability conditions of these soliton-like solutions may be analyzed within a standard variational approach [10,11]. Let us prepare, at  $z = 0$ , a Gaussian wavepacket:  $E_n(z = 0) \propto \exp \left\{ -\frac{(n-\xi_0)^2}{\gamma_0^2} + ik_0(n - \xi_0) \right\}$ . If  $\Lambda > 0$  ( $\Lambda < 0$ ), one has a soliton solution only if  $\cos k_0 < 0$  ( $\cos k_0 > 0$ ). In the following, we assume  $\Lambda > 0$  and  $\pi/2 \leq k_0 \leq \pi$  (positive velocities). For  $\gamma_0 \gg 1$ , the variational soliton-like solution should satisfy the relation:

$$\Lambda_{sol} \approx \beta_c \sqrt{\pi} \frac{|\cos k_0|}{\gamma_0}. \quad (3)$$

The long-time stability of variational solutions has been numerically checked. Hereafter, the term “solitons” denotes such variational solutions. Interestingly, discrete solitons have been experimentally observed in optical waveguides [12].

Let us consider the inhomogeneous network obtained by attaching a finite graph  $G^0$  to a single site of a chain (see Fig.1). We denote the generic waveguide with latin indices  $i, j, \dots$ . The index  $i$  can be an integer number or a greek letter  $\alpha, \beta, \dots$  according to if the waveguide  $i$  belongs to the chain or to  $G^0$  respectively. A single link connects the waveguides 0 of the chain with the waveguide  $\alpha$  of  $G^0$ . We suppose the waveguides of the infinite chain to be identical, so that their coupling terms are set to the constant value  $\beta_c$ . Soliton propagation in a generic inhomogeneous network is conveniently described within graph theory [13]. The adjacency matrix  $A^0$  of  $G^0$  is defined as:  $A_{i,j}^0 = \beta_{i,j}$  when  $i$  and  $j$  are nearest-neighbours sites of  $G^0$ , and 0 otherwise.  $G^r$  denotes the graph obtained by cutting the site  $\alpha$  from  $G^0$ , and  $A^r$  is its adjacency matrix (see Fig 1). The energy levels of  $G^0$  and  $G^r$  are the eigenvalues of  $A^0$  and  $A^r$  respectively.

The scattering of a soliton through this topological perturbation has been numerically studied in the following way. At  $z = 0$ , we prepare a Gaussian soliton, far left from 0 (i.e.,  $\xi_0 < 0$ ), moving towards  $n = 0$  ( $\sin(k_0) > 0$ ), and with a width  $\gamma_0$  related to the nonlinear coefficient  $\Lambda$  through Eq.(3). We numerically evaluate the nonlinear evolution of the electric field from Eq.(2). The group velocity is  $v = \beta_c \sin k_0$  and, when  $z_s \approx |\xi_0|/\sin(k_0)$ , the

soliton scatters through the finite graph  $G^0$ . At a position  $z$  well after the soliton scattering ( $z \gg z_s$ ), we evaluate the reflection and transmission coefficients  $\mathcal{R} = \sum_{n<0} |E_n(z)|^2$  and  $\mathcal{T} = \sum_{n>0} |E_n(z)|^2$ .

The interaction between the soliton and the defect is characterized by two length scales [14]: the soliton-defect interaction length  $z_{int} = \gamma_0/\beta_c \sin k$  and the soliton dispersion length  $z_{disp} = \gamma_0/(4\beta_c \sin(1/2\gamma_0) \cos k)$ . When  $\gamma_0 \gg 1$  and  $z_{int} \ll z_{disp}$ , the soliton is a large-fast soliton: i.e, during the scattering it may be regarded as a set of non interacting plane waves and the transmission coefficients may be computed by considering the transport of a plane wave across the topological defect. The results obtained within the linear approximation are in good agreement with the numerical solution of Eq.(2) [see Figs.3-4]. We point out that, even if a linear approximation is used [7], the nonlinearity still plays a role: it keeps together the soliton during its propagation (see Fig.2).

In the large-fast soliton regime, the momenta for perfect reflection and transmission are completely determined by the spectral properties of the graph  $G^0$  [7]. The linear eigenvalue equation to investigate is  $-\sum_m A_{n,m}E_m = \mu E_n$ , where  $A_{i,j} = \beta_{i,j}$  is the generalized adjacency matrix of the whole network. The momenta  $k$  corresponding to perfect reflection ( $\mathcal{R}(k) = 1$ ) and to perfect transmission ( $\mathcal{T}(k) = 1$ ) of a plane wave are determined by imposing the continuity at sites 0 and  $\alpha$ . One obtains  $\mathcal{R} = 1$  if  $2\beta_c \cos k$  coincides with an energy level of  $G^0$ , while  $\mathcal{T} = 1$  if  $2\beta_c \cos k$  is an energy level of the reduced graph  $G^r$  [7]. This argument can be easily extended to the situation where  $p$  identical graphs  $G^0$  are attached to  $n = 0$ . In the limit of an infinite inserted chain, the soliton propagates in a star graph, which has been recently investigated in the context of two-dimensional networks of nonlinear waveguide arrays [8].

The general analysis carried out in [7] allows for an easy identification of the graph  $G^0$  selecting the reflection (or the transmission) of a particular  $k$  (a filter). For instance, a transmission filter may be obtained by inserting a finite chain of 3 sites ( $\alpha$ ,  $\beta$  and  $\gamma$ ; see Fig.3). The momentum of perfect transmission  $k^{(0)}$  is determined by  $2\beta_c \cos k^{(0)} = -\beta_{\beta,\gamma}$ . As previously discussed, the value of the perfect transmitted momentum does not change if

$p$  identical chains are attached in  $n = 0$ , but in this case the minimum becomes sharper. In this simple case an analytical calculation, in the linear regime, of  $\mathcal{R}(k)$  gives

$$\frac{1}{\mathcal{R}} = 1 + \left( \frac{2(\beta_{\alpha,\beta}^2 + \beta_{\beta,\gamma}^2 - 2\beta_c^2) \sin(2k) - 2\beta_c^2 \sin(4k)}{p(\beta_{\beta,\gamma}^2 - 2\beta_c^2(1 + \cos(2k)))} \right)^2. \quad (4)$$

A reflection filter can be obtained by adding a finite chain of 2 sites ( $\alpha$  and  $\beta$ ). A diagonalization of  $A^0$  proves that a total reflection is obtained at the momentum  $k^{(max)}$  satisfying  $2\beta_c \cos k^{(max)} = -\beta_{\alpha,\beta}$ . The analytical expression for  $\mathcal{R}(k)$  is obtained from Eq.(4) setting  $\beta_{\beta,\gamma} = 0$  and  $p = 1$ . In Fig.3, the good agreement between numerical findings, obtained considering the full non-linear evolution, and analytical results is evidenced for large-fast solitons ( $\gamma_0 = 40$ ).

A high-pass/low-pass filter, allowing the transmission of high/low velocity solitons (see Fig.4), can also be obtained. The graph realizing a high-pass filter is given by  $p$  finite chains of length 2 and the coupling terms fixed to  $\beta_c$ . The analytical expression for  $\mathcal{R}(k)$  is obtained by setting  $\beta_{\alpha,\beta} = \beta_c$  and  $\beta_{\beta,\gamma} = 0$  in Eq.(4) (the cutoff on the momentum depends on  $p$ ). The low-pass effect is obtained with a linear chain of 3 sites ( $\alpha$ ,  $\beta$ , and  $\gamma$ ). If  $\beta_{\beta,\gamma} = 2\beta_c$ , then  $\mathcal{T} = 0$  for  $k = \pi/2$  and  $\lim_{k \rightarrow \pi} \mathcal{R} = 0$ . Therefore, one has a low-pass with the cutoff momenta depending on  $\beta_{\alpha,\beta}$  (in Fig.4  $\beta_{0,\alpha} = \beta_{\alpha,\beta} = \beta_c$  and  $\beta_{\beta,\gamma} = 2\beta_c$ ).

In conclusion, we showed that, by a pertinent choice of the topology of the graph  $G^0$ , one is able to control the reflection and the transmission of traveling kinks. The results obtained show the remarkable influence of topology on nonlinear dynamics, and apply in general to soliton propagation in discrete networks whose shape (i.e., topology) is controllable. As these results suggest, we feel that it is now both timely and highly desirable to develop the investigation of nonlinear models on general inhomogeneous networks of coupled networks, since one should expect new and interesting phenomena arising from the interplay between nonlinearity and topology.

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# FIGURES

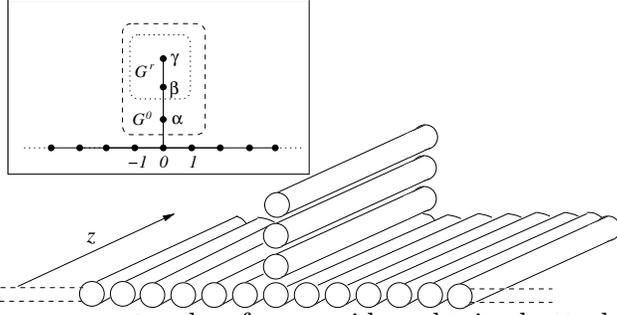


FIG. 1. The inhomogeneous networks of waveguides, obtained attaching to a chain the graph  $G^0$ , which in this case is a finite chain of length 3. In the inset we plot the corresponding graph.

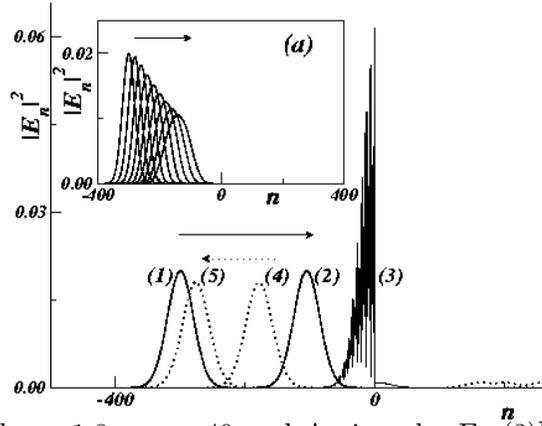


FIG. 2. A soliton [ $k_0 = 1.8$ ,  $\gamma_0 = 40$  and  $\Lambda$  given by Eq.(3)] scatters through the graph of two sites described in Fig 3. The predicted reflection coefficient is close to 1. Inset: wavepacket evolution for initial momentum  $k_0 = 0.2$  (wrong non-nonlinearity), the wavepacket spreads before hitting  $G^0$ .

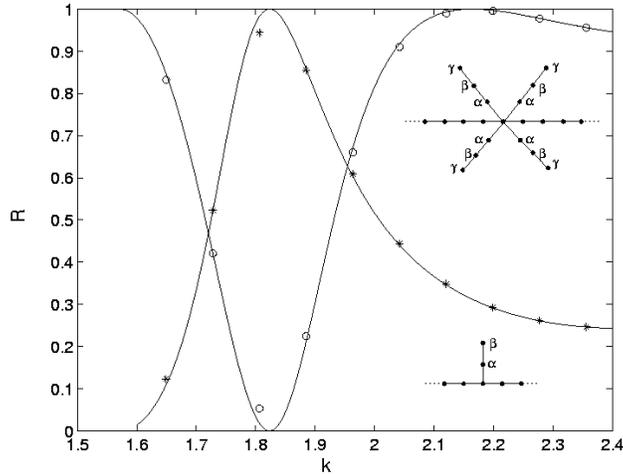


FIG. 3. The reflection coefficient vs. the momentum  $k$  for a transmission filter and a reflection filter. For the transmission filter  $\beta_{0,\alpha} = \beta_{\alpha,\beta} = \beta_c$  and  $\beta_{\beta,\gamma} = \beta_c/2$ , for the reflection filter  $\beta_{0,\alpha} = \beta_c$  and  $\beta_{\alpha,\beta} = \beta_c/2$ . Stars and circles are numerical results. The reflected and the transmitted momentum is  $\approx 1.8$ . Solid lines correspond to Eq. (4).

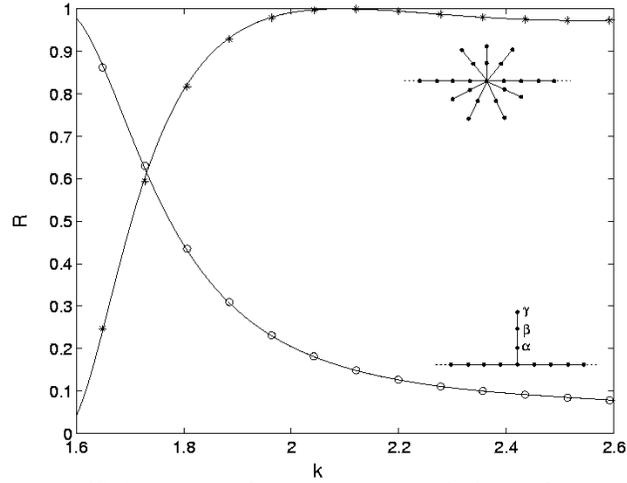


FIG. 4. The reflection coefficient vs. the momentum  $k$  for a low-pass filter and a high-pass filter (and  $p = 7$ ). Stars and circles are numerical results of Eq.(2). Solid lines corresponds to Eq.(4).