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Anyon representation of the ground-state degeneracy of the quantum frustrated XY model

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We use the extension of the Jordan-Wigner transformation to planar systems to investigate the quantum frustrated XY model for spin $\frac{1}{2}$ on a square lattice. We find that this model is mapped into the hopping Hamiltonian for a planar spinless electron in a transverse magnetic field. When we fix the frustration parameter to be a rational number such that the electron density is proportional to the total flux threading the plaquette, the model is equivalent to a Z_N gauge theory of anyons. The Z_N symmetry of the anyon theory implies a degeneracy $2N$ for the ground state of the model considered. Since we regard the frustration field as classical background, our approach has to be considered as a mean-field theory of the frustrated spin model. We briefly discuss the effect of quantum corrections to the ground-state degeneracy.

In this paper we apply the recently proposed extension of the Jordan-Wigner transformation to planar systems[1] to investigate the quantum frustrated XY model for spin $\frac{1}{2}$ on a two-dimensional square lattice. Our aim is to provide a mean-field theory description of the ground-state degeneracy of this model. In fact, we shall regard both the frustration and the Chern-Simons field introduced by the Jordan-Wigner transformation[1] as classical background.

There are many motivations for our analysis. First of all, frustrated spin Hamiltonians have been extensively used to describe the dynamics of holes in strong-coupling Heisenberg-Hubbard models for which a class of variational wave functions have been recently proposed[2] and extensive numerical work has been done.[3] In this context, the approach proposed in this paper leads to useful hints on the class of wave functions describing the hole dynamics. Another class of problems where the study of frustrated spin Hamiltonians could be relevant is provided by the theories of the fractional quantum Hall effect[4] (FQHE) and anyonic superconductivity.[5] Both phenomena are due to a novel state of matter with properties reminiscent of superfluidity.[6] We conjecture that the frustrated XY model could be regarded as a useful lattice model of the anyon superfluid. In the past spin-lattice models have been successfully proposed to explain the peculiar properties of liquid helium II.[7]

In the sequel we shall prove first that the Hamiltonian of the quantum frustrated XY model can be mapped into the hopping Hamiltonian for a planar spinless electron in a transverse magnetic field. The latter Hamiltonian has an extremely rich spectrum as shown by Hofstader, Wannier, and Azbel.[8] Recently, it has been the object of renewed interest[9,10] because of its connections with the Hartree-Fock theory of the t-j model.[2] Our results provide an elegant proof of the "molecular field" computation of Ref. 11. Second, following the constructions of Refs. 1, 9, and 12, we show that the quantum frustrated XY model can be regarded as a theory of anyons[13] on the lattice interacting with a Z_N gauge potential when the frustration parameter f is a rational number. The statistical parameter of the anyons θ is related to the frustration by $\theta=2\pi f$. Our analysis confirms and extends the results of Ref. 14, allowing one to interpret the uniform frustration field as a Chern-Simons gauge field. The scenario is similar to the one advocated by Girvin for the mean-field theory of the FQHE.[4]

For anyons on a torus, the Z_N gauge symmetry has been analyzed in Refs. 15 and 16.

The model Hamiltonian we posit to describe the quantum frustrated XY model is

$$H=J\sum_{\langle i,j\rangle}[\mathbf{S}_i\cdot U_{ij}\cdot\mathbf{S}_j]\,. \tag{1}$$

In Eq. (1), i,j denote arbitrary lattice sites, $\langle i,j\rangle$ is the nearest-neighbor sum, and U_{ij} is a link gauge degree of freedom describing the frustration field.[17] In the following we consider the ease in which J is a ferromagnetic coupling.[2,7] The extension to frustrated antiferromagnets goes through with only minor modifications.

Equation (1) is required to be invariant under the U(1) gauge transformation

$$S_i\to\Lambda_i S_i\,, \tag{2a}$$

$$U_{ij}\to\Lambda_i U_{ij}\Lambda_j^*\,, \tag{2b}$$

with $\Lambda_i\equiv\exp i\theta_i$.

As a consequence of gauge invariance the partition function has to depend only on the gauge-invariant quantity $\prod_p U_{ij}$ constrained to equal

$$\prod_p U_{ij}\equiv U_{ij}U_{jk}U_{ke}U_{ei}=e^{-2\pi i f}\,. \tag{3}$$

In Eq. (3) f is the frustration parameter. It measures the ratio between the flux threading the plaquette and the unit magnetic flux $\phi_0=e/hc$. Due to the invariance of the theory under $f\to f+n\,(n\in Z)$ and $f\to -f$ we have $0\le f\le\frac{1}{2}$.

In its classical version, the Hamiltonian (1) provides an elegant description of Josephson-junction arrays in a transverse magnetic field.[18] Via a Villain transformation,[19] the classical Hamiltonian can be mapped into a lattice plasma model of the form

$$H=+\pi\tilde{J}\sum_{r,r'}[m(r)+f]G(r,r')[m(r')+f]\,. \tag{4}$$

In Eq. (4) r and r' denote arbitrary points on the dual lattice, $G(r-r')\rightarrow \log|r-r'|$ as $|r-r'|\rightarrow\infty$, and the charges $m(r)$ are restricted to integer values. The average value of the plasma charges over the entire lattice is constrained to equal f.

To fermionize the spin system one notices that the SO(3) Lie algebra with irreducible $s=\frac{1}{2}$ representation can be realized by the fermionic variables ψ_i and $\psi_i^\dagger$ with $\{\psi_i,\psi_j^\dagger\}=\delta(i-j)$ and all other anticommutators vanishing as

$$S_i^- = \exp\left[+i(2k+1)\sum_j \Theta(i-j)J_0(j)\right]\psi_i\,, \tag{5a}$$

$$S_i^+ = \psi_i^\dagger \exp\left[-i(2k+1)\sum_j \Theta(i-j)J_0(j)\right]\,, \tag{5b}$$

with $S_i^\pm = S_{i2}\pm S_{i2}$ and $J_0(j)\equiv\psi_j^\dagger\psi_j$ equal to 0 or 1.

The fermionic version of (1) is then

$$H=\frac{J}{2}\sum_{\langle i,j\rangle}(\psi_i^\dagger e^{i\bar{A}_i(j)}\psi_j+\text{H.c.})\,, \tag{6}$$

with

$$\bar{A}_i(j)=A_i(j)+(2k+1)\sum_z[\Theta(j+i-z) -\Theta(j-z)]J_0(z)\,. \tag{7}$$

In Eqs. (6) and (7) $\Theta(i-j)$ is the angle between the vector connecting sites i and j of the lattice and a reference direction. Since in a quantum theory J_0 has eigenvalues 0 and 1 the ambiguity of 2π in the definition of Θ has no effect in Eqs. (6) and (7). Furthermore, k is an integer, $\Theta(0)=0$, and $A_i(j)$ is the phase of the field U_{ij}.

In Eq. (7) the total gauge potential is the sum of the frustration field and a Chern-Simons field,[20] both of which we regard here as classical background fields.

The electron hopping model described by Eq. (6) may be derived from an action with a Chern-Simons term whose coefficient is $\theta=\pi(2k+1)$. The flux through the plaquette of the Chern-Simons field equals $\pi(\text{mod}\,2\pi)$.

The total flux threading the plaquette is constrained to equal

$$\sum_P \bar{A}_i(j)=2\pi(\tfrac{1}{2}-f)\equiv\Phi\,. \tag{8}$$

Since we require the model described by (6) to be periodic in $\bar{A}_i(j)$ [i.e., $\bar{A}_i(j)$ and $\bar{A}_i(j)=\bar{A}_i(j)+2\pi n_j$ with n_j arbitrary integer cannot be distinguished] time-reversal invariance is broken except for the special cases $\Phi=0,\pi$. These cases correspond to the fully frustrated ($f=\frac{1}{2}$) and the ordinary ($f=0$) quantum XY model, respectively. For intermediate rational values of the frustration the ground state of (6) is expected to be a chiral spin state.[21] We recall that f has to be a rational number when one imposes doubly periodic boundary conditions on the lattice. In the following, we shall restrict to only such values of f.

In Ref. 10 it has been shown that the energy of the chiral spin state reaches an absolute minimum when the flux per plaquette equals the electron density per site, i.e., when

$$\langle J_0\rangle=\frac{2\pi}{N}\sum_i J_0(i)=\Phi\,. \tag{9}$$

Equation (9) is equivalent to the requirement of charge neutrality of the Coulomb gas representation of the frustrated spin system and is satisfied by the gauge-invariant states of the quantum frustrated XY model. As a consequence, this model can be mapped into a theory of anyons on the lattice interacting with a Z_N gauge potential.

To construct the anyonic model one notices[9,12] that if a flux Φ threads a plaquette adjacent to a fermion, the particle with the flux (i.e., the anyon) obeys θ statistics,[13]

$$\theta=\pi-\Phi=2\pi f\,. \tag{10}$$

In Eq. (10) θ is the statistical parameter of the anyon and f is the frustration parameter. The ensuing Hamiltonian for the anyons is

$$H=\sum_{\langle ij\rangle}\phi_i e^{i\theta_{ij}}\phi_j^\dagger\,, \tag{11}$$

where $\theta_{ij}=2\pi fN$ with N the smallest integer such that Nf is an integer.

Notice that for $f=0$ Eq. (11) provides a free theory of hard-core bosons. This corresponds to the exact ground state of the quantum XY model[22] for spin $\frac{1}{2}$. For $f=0$ Eq. (11) provides a description of the frustrated spin system in term of free fermions with flux π per plaquette. The ground state is in this case the so-called Affleck-Marston flux phase.[23] The case $f=\frac{1}{4}$ corresponds to semions[24] in the proposed mean-field approximation.

The discrete symmetry Z_N of the anyonic theory implies a degeneracy $2N$ for the ground state of the quantum frustrated XY model. For the classical model the ground-state degeneracy has been investigated numerically for $f=\frac{1}{2}$ and $f=\frac{1}{3}$.[18,25]

As pointed out in Refs. 15 and 26, the origin of the discrete symmetry Z_N is topological. This is easily seen if one notices that since the statistical parameter of the anyons is $2\pi f$ the pertinent one-dimensional unitary representation of the braid group is given by

$$\chi[\sigma^j]=e^{i\theta\pi},\quad \chi[(\sigma^j)^{-1}]=e^{-i\theta\pi}\forall_j \tag{12}$$

with $\theta=2f$. When $\theta=M/N$ with M and N prime integers there are N distinct phases and the representation is finite dimensional even though the group itself has infinite elements. The resulting N species of anyons are thus degenerate in energy. Due to Eqs. (5) to the symmetry $f\rightarrow -f$ the Z_N symmetry of the anyonic theory implies that the mean-field theory ground state of the quantum frustrated XY model is $2N$ degenerate.

The above argument—together with the computation of the ground-state energy of the classical frustrated XY model[18]—leads to the conjecture that irrational values of f should be treated as a sort of decompactification limit ($N\rightarrow\infty$)—analogous to the one of Ref. 27—of the theory with rational frustration. We recall that f is constrained to equal a rational number when the boundary conditions on the lattice are doubly periodic. The elusive precise definition of this limiting procedure relevant for many physical applications is the object of ongoing investigation.

Our approach treats both the frustration and the Chern-Simons field as classical background fields; thus it has to be regarded as a mean-field theory of the quantum frustrated XY model. It would be interesting to compute the effects of quantum fluctuations at least on the ground-state degeneracy of the frustrated XY model. Our analysis reduces this problem to the evaluation of the quantum corrections to the statistical parameter of anyons on the lattice. In the continuum theory, when the matter fields coupled minimally with a Chern-Simons field are massive, the statistical parameter is not subject to renormalization.[28] Thus, an answer to this question depends upon the specific choice of the U_{ij}'s. Explicit model computations are now in progress.

A merit of our approach is that it provides a simple constructive argument by which the internal frustration field can be regarded as a Chern-Simons field. The emerging scenario is reminiscent of the Landau-Ginzburg theory of the FQHE.[4] It is natural to ask if the magnetic model we considered in this paper may be considered as a spin-lattice model of quantum Hall liquid. We shall report on the results of the computation of the magnetic susceptibility in a separate paper.

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