Physics Letters A 161 (1991) 41–44
North-Holland

PHYSICS LETTERS A

SU(1, 1) spectrum generating algebra for the quantum damped harmonic oscillator

Vincenzo Barone
Dipartimento de Fisica e Sezione I.N.F.N., Università di Perugia, 06100 Perugia, Italy

Vittorio Penna [1] and Pasquale Sodano [2]
Center for Nonlinear Studies, Los Alamos National Laboratory, Los Alamos, NM 87545, USA

Received 23 September 1991; accepted for publication 17 October 1991
Communicated by D.D. Holm

Using the spectrum generating algebra method, we find the complete set of exact eigenstates for the quantum damped harmonic oscillator. The states which diagonalize our quantum mechanical model Hamiltonian are the Lindblad–Nagel states which provide an unitary irreducible representation of the SU(1, 1) algebra. We derive an integral representation of the Lindblad–Nagel states in terms of SU(1, 1) generalized coherent states. We discuss possible applications of this formula.

Spectrum generating algebras (SGA) in quantum physics have for a long time been a simple and powerful tool in characterizing and describing the dynamics of physical systems (for recent reviews of the subject see ref. [1]). By SGA we mean an algebra with generators that can be used to replace the canonical variables in the Hamiltonian, thus allowing for a purely algebraic derivation of its spectrum. In this approach, unitary irreducible representations of a given SGA provide the exact eigenstates of the Hamiltonian.

In this Letter, we apply the SGA method to the quantum damped harmonic oscillator (QDHO). The model Hamiltonian we consider is the one introduced by Bateman [2] and Feshbach and Tikochinsky [3] to analyse the classical and quantum properties of the DHO. We shall show that this Hamiltonian has an SU(1, 1) SGA and that its eigenstates are the Lindblad–Nagel (LN) states [4]. Our analysis thus provides an independent proof of the statement that dissipative quantum theories have the non-compact group SU(1, 1) as a relevant symmetry [5,6]. On the space of the LN states the quantum mechanical Hamiltonian is a Hermitian operator.

In this Letter, we shall also derive an integral representation of the LN states in terms of SU(1, 1) generalized coherent states [7]. As a byproduct, we shall obtain an integral equation relating the LN states to the highest weight states of the standard representation of SU(1, 1). Since the representation we obtain is generic to Hamiltonians which can be expressed as a linear superposition of the Casimir operator and a non-compact generator of SU(1, 1), it is useful also to describe the eigenstates for quantum models defined on manifolds of constant negative curvature [8]. We shall report on this application in a separate paper.

In order to provide a consistent Hamiltonian formulation of the QDHO, one needs at least two degrees of freedom to describe the system [2–6]. Our choice of the Hamiltonian for a QDHO with mass m, elastic constant k and friction coefficient γ is [2,3]

$$H=\frac{1}{m}p_x p_y+m\Omega^2 xy+\frac{\gamma}{2m}(p_y y-p_x x)\,, \tag{1}$$

with $\Omega^2=(1/m)(k-\gamma^2/4m)$. The extra degree of

[1] Permanent address: Dipartimento di Fisica, Politecnico di Torino, Corso Duca degli Abbruzzi 24, 10129 Turin, Italy.
[2] Permanent address: Dipartimento di Fisica e Sezione I.N.F.N., Università di Perugia, 06100 Perugia, Italy.

freedom described by (y, p_y) represents an enhanced harmonic oscillator [2,3].

Introducing the annihilation operators a and b,

$$a=(2m\hbar\Omega)^{-1/2}(m\Omega x+\mathrm{i}p_x)\ , \tag{2a}$$

$$b=(2m\hbar\Omega)^{-1/2}(m\Omega y+\mathrm{i}p_y)\ , \tag{2b}$$

the Hamiltonian (1) becomes

$$H=\hbar\Omega(ab^++ba^+)$$
$$-\tfrac{1}{2}\mathrm{i}\Gamma\hbar(b^2-b^{+2}+a^{+2}-a^2)\ , \tag{3}$$

with $\Gamma=\gamma/2m$.

In this realization the kinetic energy mixes the two degrees of freedom whereas the dissipative part is written in a separable form.

One can easily recognize in eq. (3) the non-compact generator and the Casimir operator of the SU(1, 1) algebra in the following two-mode realization,

$$D_1=\tfrac{1}{4}(b^2+b^{+2}-a^2-a^{+2})$$
$$=\frac{1}{4m\hbar\Omega}[p_x^2-p_y^2-m^2\Omega^2(x^2-y^2)]\ , \tag{4a}$$

$$D_2=\frac{1}{4\mathrm{i}}(b^{+2}-b^2+a^2-a^{+2})$$
$$=-\frac{1}{2\hbar}(yp_y-xp_x)\ , \tag{4b}$$

$$D_3=\tfrac{1}{2}(a^+a+b^+b+1)$$
$$=\frac{1}{4m\hbar\Omega}[p_x^2+p_y^2+m^2\Omega^2(x^2+y^2)]\ , \tag{4c}$$

$$D_4=\tfrac{1}{2}(ab^++a^+b)$$
$$=\frac{1}{2\hbar}(m\Omega xy+p_xp_y/m\Omega)\ . \tag{4d}$$

The commutation relations for the D_i are

$$[D_1, D_2]=-\mathrm{i}D_3\ , \tag{5a}$$

$$[D_2, D_3]=\mathrm{i}D_1\ , \tag{5b}$$

$$[D_3, D_1]=\mathrm{i}D_2\ , \tag{5c}$$

$$[D_4, D_i]=0,\quad i=1, 2, 3\ , \tag{5d}$$

while the Casimir operator C is related to D_4 by

$$C=D_3^2-D_1^2-D_2^2=D_4^2-\tfrac{1}{4}\ . \tag{5e}$$

As a consequence of eqs. (4), (5), the Hamiltonian is rewritten as

$$H=2\hbar(\Omega D_4-\Gamma D_2)\ . \tag{6}$$

In order to find explicit eigenstates of (6), it is convenient to perform a unitary transformation $R_{\pi/2}=\exp(\tfrac{1}{2}\mathrm{i}\pi W)$, on the D_i induced by the operator $W=(1/2\mathrm{i})(ab^+-a^+b)$. As a result, the set of D_i changes into a set of new operators k_i defined by

$$k_1\equiv R^+_{\pi/2}D_1R_{\pi/2}=\tfrac{1}{2}(ab+a^+b^+)\ , \tag{7a}$$

$$k_2\equiv R^+_{\pi/2}D_2R_{\pi/2}=\frac{1}{2\mathrm{i}}(a^+b^+-ab)\ , \tag{7b}$$

$$k_3\equiv R^+_{\pi/2}D_3R_{\pi/2}=\tfrac{1}{2}(N_a+N_b+1)\ , \tag{7c}$$

$$k_4\equiv R^+_{\pi/2}D_4R_{\pi/2}=\tfrac{1}{2}(N_a-N_b)\ , \tag{7d}$$

with

$$N_a=a^+a,\quad N_b^+=b^+b\ . \tag{7e}$$

In terms of the new operators k_i the Hamiltonian (6) reads as

$$H\equiv H_0+H_{\mathrm{int}}=2\hbar\Omega k_4-2\hbar\Gamma k_2$$
$$=\hbar\Omega(a^+a-b^+b)+\mathrm{i}\hbar\Gamma(a^+b^+-ab)\ . \tag{8}$$

A few remarks are now in order. Even if we set $\Gamma=0$, eq. (8) describes a system with an energy spectrum not bounded from below. This is not surprising since we are dealing with a system in which dissipation causes a continuum energy exchange between the a and b degrees of freedom. A similar feature is shared by the dissipative Hamiltonian in thermo field dynamics (TFD) [6]. Notice also that H is invariant under time reversal, since under this discrete symmetry operation $a\to a^+$, $b\to b^+$, and $\Gamma\to-\Gamma$. Finally, the creation of a (b) quanta does not imply a change in the eigenvalue of H_0 if the same number of b (a) quanta is simultaneously created. It is easy to convince oneself that the rotation induced by the operator W corresponds to a canonical transformation mapping (1) into

$$H=\frac{1}{2m}(p_x^2-p_y^2)+\tfrac{1}{2}m\Omega^2(x^2-y^2)$$
$$+\Gamma(xp_y+yp_x)\ . \tag{9}$$

To find the eigenstates of (8) we first diagonalize k_3 and k_4. Choosing the standard basis $|l, k\rangle$ we have

$$k_3|l,k\rangle=k|l,k\rangle\ , \tag{10a}$$

$$k_4|l,k\rangle=\tfrac{1}{2}(2l+1)|l,k\rangle\ , \tag{10b}$$

with k and l determined by

$$n_a=k-l-1\ , \tag{11a}$$

$$n_b=k+l\ . \tag{11b}$$

In eqs. (11) n_a (n_b) is the eigenvalue of N_a (N_b) and k is the eigenvalue of k_3. The unitary irreducible representations of SU(1, 1) have been classified long ago by Bargmann [9]. There are two discrete principal series – T_l^+ and T_l^- – which have l negative integer or half integer and $k\geqslant -l$ for T_l^+ while $k\leqslant l$ for T_l^-. Since n_a, $n_b\geqslant 0$ one must use the T_l^+ series. The highest weight state is $|l,-l\rangle\equiv|l,|l|\rangle$ corresponding to $n_a=2l+1$ and $n_b=0$. The state with $n_a=n_b=0$ is associated to $l=-\frac{1}{2}$. We shall denote this state with $|0\rangle$.

To diagonalize (8), we need the eigenstates of the non-compact operators k_2 and k_4. This problem has been solved by Lindblad and Nagel [4] who found a complete set of states labelled by a real number Λ satisfying

$$k_2|l,\Lambda\rangle=\Lambda|l,\Lambda\rangle\ , \tag{12a}$$

$$k_4|l,\Lambda\rangle=\tfrac{1}{2}(2l+1)|l,\Lambda\rangle\ . \tag{12b}$$

The LN states can be expressed in terms of the vectors of the standard basis as

$$|l,\Lambda\rangle=S(l,\Lambda)\sum_{k=|l|}^{\infty}\left(\frac{\Gamma(k-l)}{\Gamma(k+l+a)}\right)^{1/2}\times\int_{-1}^{1}\frac{\mathrm{d}t}{(1-t^2)^{1-|l|}}\,t^{k-|l|}\times\exp\left[-\mathrm{i}\Lambda\ln\left(\frac{1+t}{1-t}\right)\right]|l,k\rangle\ , \tag{13}$$

with $S(l,\Lambda)$ a normalization coefficient determined by

$$\langle l,\Lambda'|l,\Lambda\rangle=\delta(\Lambda'-\Lambda)\ . \tag{14}$$

The LN states (12) provide the set of exact eigenstates of (8). Notice that the eigenstates associated to $l=-\frac{1}{2}$ correspond to $n_b-n_a=0$ and define the zero mode space of $H_0(k_4)$. We denote this infinite degenerate space by $|\Lambda\rangle$. In the following, we shall prove that all the states $|\Lambda\rangle$ can be obtained as functionals of the state $|0\rangle$ belonging to the standard representation of SU(1, 1). From a formal point of view – here we consider a quantum mechanical system with only two degrees of freedom – the state $|\Lambda\rangle$ is strikingly similar to the "thermal vacuum" in TFD [6]. On the space of LN states (8) is a Hermitian operator.

To make explicit the functional relation between $|\Lambda\rangle$ and $|0\rangle$, we shall prove that a LN state can be written in terms of SU(1, 1) generalized coherent states as

$$|l,\Lambda\rangle=S'(l,\Lambda)\int_{-\infty}^{\infty}\mathrm{d}s\,\exp(-\mathrm{i}s\Lambda)|s\rangle\ , \tag{15}$$

with

$$|s\rangle\equiv\exp[\tfrac{1}{2}s(k_+-k_-)]|l,|l|\rangle$$
$$=\exp(\mathrm{i}sk_2)|l,|l|\rangle\ . \tag{16}$$

In eq. (16) $k_\pm=k_1\pm\mathrm{i}k_2$. The advantage of (15) lies in the fact that it provides a rather simple integral representation of an arbitrary LN state in terms of the coherent state $|s\rangle$. For $l=-\frac{1}{2}$ we obtain

$$|\Lambda\rangle=S'(\Lambda)\int_{-\infty}^{\infty}\mathrm{d}s\,\exp[\mathrm{i}s(k_2-\Lambda)]|0\rangle\ . \tag{17}$$

The state $|\Lambda\rangle$ due to (17) is understood as a state with neither vanishing or definite occupation numbers of a and b but rather as a superposition of states $|0(s)\rangle=\exp(\mathrm{i}sk_2)|0\rangle$ each having the same number of a and b quanta. Notice that the unitary operator $U(s)=\exp(\mathrm{i}sk_2)$ commutes with H.

In order to prove eq. (15), we recall that the SU(1, 1) generalized coherent states are obtained by the action of a displacement operator on the highest weight vector of the standard basis

$$|\eta\rangle=D(\eta)|l,|l|\rangle$$
$$=\exp(\eta k_+-\eta^*k_-)|l,|l|\rangle\ . \tag{18}$$

In eq. (18) $k_\pm=k_1\pm\mathrm{i}k_2$ and η is a complex number. Introducing a new parameter ζ ($\zeta=\mathrm{tgh}|\eta|\mathrm{e}^{\mathrm{i}\varphi}$), use of the Baker–Campbell–Hausdorff formula allows us to rewrite (18) as [10]

$$|\zeta\rangle=(1-|\zeta|^2)^{|l|}\exp(\zeta k_+)|l,|l|\rangle\ . \tag{19}$$

Since

$$|l,k\rangle = \frac{\Gamma(k+|l|)}{(k-|l|)!\Gamma(2l)} k_{+}^{k-|l|} |l,|l|\rangle \tag{20}$$

expanding the exponential in (19) leads to

$$|\zeta\rangle = (1-|\zeta|^2)^{|l|} \times \sum_{k=|l|}^{\infty} \left(\frac{\Gamma(|l|+k)}{(k-|l|)!\Gamma(2l)}\right)^{1/2} \zeta^{k-|l|} |l,k\rangle . \tag{21}$$

Eq. (21) allows one to write the generic LN state as

$$|l,\Lambda\rangle = S'(l,\Lambda) \times \int_{-1}^{1} \frac{dt}{1-t^2} \exp\left[-i\Lambda \ln\left(\frac{1+t}{1-t}\right)\right] |t\rangle , \tag{22}$$

where $|t\rangle$ is a coherent state labelled by the real number t. Upon defining a new parameter s as $s=2\ln[(1+t)/(1-t)]^{1/2}$ eq. (15) is easily obtained.

It is easy to derive the extension of eq. (15) for the three classes of SU(1, 1) unitary irreducible representations. As pointed out earlier, this formula shows itself very useful to describe the eigenstates of quantum Hamiltonian models with SU(1, 1) symmetry such as the ones defined as manifolds of constant negative curvature [8]. From a more formal standpoint, the integral representation (15) could provide a tool to investigate the structure of the raising and lowering operators for the LN generalized eigenvectors space [4].

In this paper we reported on the exact diagonalization of the Hamiltonian (1) and derived the integral representation (15) for the LN states. Unfortunately, the model Hamiltonian introduced in refs. [2,3] provides – in our opinion – only a formal description of the QDHO since the separation of the degrees of freedom associated with the "physical" and the "image" oscillators is very difficult to achieve without very strong ad hoc assumptions [3,11]. We believe that the approach of TFD [6] leads to a more transparent physical theory. In the context of TFD, the integral representation (15) provides a nice expression for the quantum mechanical analogue of the "thermal vacuum." Work to further elucidate the structure of the zero mode space of H_0 – as well as to determine the pertinent raising and lowering operators acting on this space – is now in progress.

Two of us (V.P. and P.S.) thank the Center for Nonlinear Studies (C.N.L.S.) of Los Alamos National Laboratory for providing the stimulating environment apt to start a scientific collaboration. During the course of our work, we enjoyed discussions with H. Umezawa, M. Rasetti, Y. Yamanaka and G. Vitiello. P.S. is grateful to N.A.T.O. and I.N.F.N. while V.P. thanks the "Fondazione Angelo della Riccia" for partial financial support during residence at the C.N.L.S.

References

[1] A. Bohm, in: Proc. XVIIIth Int. Colloq. on Group theoretical methods in physics, eds. V.I. Manko et al. (Springer, Berlin, 1991);
Y. Ne'eman, in: Proc. XVIIIth Int. Colloq. on Group theoretical methods in physics, eds. V.I. Manko et al. (Springer, Berlin, 1991);
F. Iochello, in: Proc. XVIIIth Int. Colloq. on Group theoretical methods in physics, eds. V.I. Manko et al. (Springer, Berlin, 1991).

[2] H. Bateman, Phys. Rev. 38 (1931) 815.

[3] H. Feshbach and Y. Tikochinsky, Trans. NY Acad. Sci. Series 2, Vol. 38. Festschrift for I.I. Rabi (1977) p. 44.

[4] G. Lindblad and B. Nagel, Ann. Inst. H. Poincaré 13 (1970) 27.

[5] E. Celeghini, M. Rasetti, M. Tarlini and G. Vitiello, Mod. Phys. Lett. B 3 (1989) 1213;
E. Celeghini, M. Rasetti and G. Vitiello, to be published in Ann. Phys. (NY);
E. Celeghini, M. Rasetti and G. Vitiello, in: Thermal field theory, eds. H. Ezawa, T. Arimitsu and Y. Hashimoto (North-Holland, Amsterdam 1991) p. 189.

[6] H. Umezawa and Y. Yamanaka, Adv. Phys. 37 (1988) 531; Phys. Lett. A 155 (1991) 75.

[7] M. Rasetti, Int. J. Theor. Phys. 13 (1975) 425;
A.M. Perelomov, Commun. Math. Phys. 26 (1972) 222.

[8] A. Comtet, Ann. Phys. 173 (1987) 185;
C.G. Callan and F. Wilczek, Nucl. Phys. B 340 (1990) 366.

[9] V. Bargmann, Ann. Math. 48 (1947) 568.

[10] A.M. Perelomov, Generalized coherent states and their applications (Springer, Berlin, 1986) p. 73.

[11] H. Dekker, Phys. Rep. 80 (1981) 1.