

# Variational Method for Studying Solitons in the Korteweg-DeVries Equation

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## Abstract

We use a variational method based on the principle of least action to obtain approximate time-dependent single soliton solutions to the KdV equation. A class of trial variational functions of the form  $u(x, t) = -A(t) \exp[-\beta(t) |x - q(t)|^{2n}]$ , with  $n$  a continuous real variable, is used to parametrize time-dependent solutions. We find that this class of trial functions leads to soliton-like solutions for all  $n$ , moving with fixed shape and constant velocity, and with energy and mass conserved. Minimizing the energy of the soliton with respect to the

parameter  $n$ , we obtain a variational solution that gives an extremely accurate approximation to the exact solution.

## 1 Introduction

In a previous work [1], we introduced a post-Gaussian variational approximation, a continuous family of trial variational functions more general than Gaussians, which can still be treated analytically. We demonstrated their usefulness by discussing several features of the general nonlinear Schrodinger equation: we derived an approximation to the one-dimensional soliton solution and showed the “universality” of the critical exponent for blowup in the supercritical case; for the critical case we calculated the critical mass necessary for blowup.

In the present work we consider another application of the post-Gaussian variational method, deriving an approximation to the one-dimensional KdV soliton. Of course the soliton solution to this exactly integrable system is well-known and easy to find by direct integration. The aim here is rather to provide another illustration of the power, generality, and accuracy of our method in a problem with some new features, in preparation for treating other more difficult problems.

The time-dependent variational principle [2] is very useful for quantum mechanics [3] and field theory problems as well as for studying nonlinear systems described by a Lagrangian [4]. In our recent work we showed how one could obtain information about solitons and self-focusing (blow-up) [1], [4] in the nonlinear Schrodinger equation using a version of Dirac’s variational principle and a canonical set of variational parameters. This choice of parameters led automatically to a conserved energy for the effective Lagrangian. For the KdV equation, however, a similar choice of variational trial functions leads to a Lagrangian that is not automatically in canonical form. We employ the method of Faddeev and Jackiw [5] to explicitly obtain a canonical Lagrangian with symplectic structure, which leads to a conserved energy for the variational parameters as well as a Hamiltonian structure. Assuming a variational ansatz of the form  $u(x, t) = -A(t) \exp[-\beta(t) |x - q(t)|^{2n}]$ , we find that the conserved energy of the effective Lagrangian is minimized for  $n = 0.877$ , giving an extremely accurate energy, velocity, and shape for the approximate single soliton solution.

## 2 Action for the KdV Equation and the Variational Principle

The starting point for the derivation of the KdV equation is the action

$$\Gamma = \int L dt, \quad (1)$$

where  $L$  is given by [6]

$$L = \int \left( \frac{1}{2} \varphi_x \varphi_t - (\varphi_x)^3 - \frac{1}{2} (\varphi_{xx})^2 \right) dx. \quad (2)$$

From this Lagrangian, one can determine the conserved Hamiltonian

$$H = \int [(\pi \dot{\varphi}) - L] dx = \int \left[ (\varphi_x)^3 + \frac{1}{2} (\varphi_{xx})^2 \right] dx, \quad (3)$$

where  $\pi \equiv \frac{\delta L}{\delta \dot{\varphi}}$ . Because the action is stationary with respect to variation in  $\varphi$  (i.e.  $\delta \Gamma / \delta \varphi = 0$ ), we obtain the equation

$$\varphi_{xt} - 6\varphi_x \varphi_{xx} + \varphi_{xxxx} = 0, \quad (4)$$

which we recognize as the KdV equation with the identification  $u \equiv \varphi_x$ :

$$u_t - 6uu_x + u_{xxx} = 0, \quad (5)$$

$\Gamma$  is the action of the system described by  $\varphi(x, t)$ ; the variational principle is a version of Hamilton's least-action principle. In the variational principle  $\varphi$  is an arbitrary square integrable function. To obtain an approximate solution to the KdV equation, we consider a restricted class  $\varphi_v(x, t)$ , constrained to a form that tries to capture the known behavior of the full  $\varphi$  for the problem at hand. In this article we will consider only trial wave functions that are capable of describing the motion of an initial configuration which can be reasonably approximated at time  $t = 0$  by  $u(x, 0) = -A \exp[-\beta|x - a|^{2n}]$  for suitable choice of  $A, \beta, a$  and  $n$ . Thus, we choose for our trial wave function

$$u_v(x, t) = -A(t) \exp[-\beta(t)|x - q(t)|^{2n}], \quad (6)$$

where  $n$  is an arbitrary continuous, real parameter.

The variational parameters have a simple interpretation in terms of expectation values with respect to the “probability”

$$P(x, t) = \frac{[u_v(x, t)]^2}{M(t)}, \quad (7)$$

where the mass  $M$  is defined as

$$M(t) \equiv \int [u_v(x, t)]^2 dx. \quad (8)$$

(Here we allow  $M$  to be a function of  $t$ , even though  $M$  is conserved for the exact KdV equation.)

Since  $\langle x - q(t) \rangle = 0$ ,  $q(t) = \langle x \rangle$ . From (8) we have

$$A(t) = \frac{M^{1/2}(2\beta)^{1/4n}}{\left[2\Gamma\left(\frac{1}{2n} + 1\right)\right]^{1/2}}. \quad (9)$$

The inverse width  $\beta$  is related to

$$G_{2n} \equiv \langle |x - q(t)|^{2n} \rangle = \frac{1}{4n\beta}. \quad (10)$$

Thus, we can write

$$u = \varphi_x = -\frac{M^{1/2}(2\beta)^{1/4n}}{\left[2\Gamma\left(\frac{1}{2n} + 1\right)\right]^{1/2}} \exp\left[-\beta(t)|x - q(t)|^{2n}\right]. \quad (11)$$

To evaluate the action we first must determine  $\varphi$ . A convenient choice of integration constant gives

$$\begin{aligned} \varphi(x, t) &= -A(t) \int_{q(t)}^x \exp\left[-\beta(t)|y - q(t)|^{2n}\right] dy \\ &= -A(t)(2n)^{-1} \beta^{-1/2n} \epsilon(x - q(t)) F\left[\beta^{1/2n}|x - q(t)|\right], \end{aligned} \quad (12)$$

where

$$F\left[\beta^{1/2n}(x - q(t))\right] = \int_0^{\beta^{1/2n}|x - q(t)|^{2n}} e^{-z} z^{\frac{1}{2n}-1} dz. \quad (13)$$

The sign function  $\epsilon(x)$  explicitly displays the oddness properties of  $\varphi$ :

$$\varphi(x - q(t), t) = -\varphi(-[x - q(t)], t). \quad (14)$$

For  $n = 1$  we have

$$F[\beta^{1/2}(x - q(t))] = \pi^{1/2} \text{Erf}[\beta^{1/2}(x - q(t))]. \quad (15)$$

The oddness property of  $\varphi$  guarantees conservation of  $M$  and  $\beta$ , as we will see below. (For a two soliton ansatz neither  $M$  nor  $\beta$  would be time-independent.)

We next evaluate the action, given by (1) and (2), for the trial wave function in (6). First consider

$$\varphi_t = -\dot{q}u + \left( \frac{\dot{M}}{2M} + \frac{\dot{\beta}}{4n\beta} \right) \varphi + \dot{\beta}G(x - q(t)), \quad (16)$$

where

$$G(x - q(t)) \equiv A \int_0^{x-q(t)} |y|^{2n} \exp[-\beta(t)|y|^{2n}] dy. \quad (17)$$

We notice that  $G$  is also odd in  $x - q(t)$ . Thus, in determining  $\int (\frac{1}{2}\varphi_x\varphi_t) dx$ , only the first term of (16) contributes and we obtain  $-M\dot{q}/2$  for the resulting integral. This oddness property causes  $\beta$  to be a constraint rather than a dynamical variable in the ensuing Lagrangian dynamics.

Evaluating the other terms in (2), we obtain

$$\begin{aligned} \Gamma(q, \beta, M, n) &= \int \left( -\frac{1}{2}M\dot{q} + C_1(n)\beta^{1/4n}M^{3/2} - C_2(n)M\beta^{1/n} \right) dt \\ &\equiv \int L_1(q, \dot{q}, M, \beta) dt, \end{aligned} \quad (18)$$

where

$$\begin{aligned} C_1(n) &= \left( \frac{8}{9} \right)^{1/4n} \left[ 2\Gamma\left( \frac{1}{2n} + 1 \right) \right]^{-1/2} \\ C_2(n) &= \frac{1}{4}n2^{1/n} \frac{\Gamma\left( 2 - \frac{1}{2n} \right)}{\Gamma\left( \frac{1}{2n} + 1 \right)}. \end{aligned} \quad (19)$$

The variational equations  $\delta\Gamma/\delta q_i = 0$  yield a set of equations for the parameters  $q, \beta$  and  $M$ . However it is more useful to put our first Lagrangian  $L_1$  into canonical form, so that the conserved Hamiltonian can be displayed and the *unconstrained* Lagrangian can be obtained. Following the work of Fadeev and Jackiw [5], we first rewrite  $L_1$  as

$$L_2 = \frac{1}{4} (q\dot{M} - \dot{q}M) - H_2(\beta, M), \quad (20)$$

where

$$H_2(\beta, M) = -C_1(n)\beta^{1/4n}M^{3/2} + C_2(n)M\beta^{1/n}. \quad (21)$$

We next recognize that  $\beta$  is a variable of constraint since  $\dot{\beta}$  does not appear in  $L_2$ . We eliminate  $\beta$  (using  $\delta\Gamma/\delta\beta = 0$ ) and find

$$\beta = [d(n)]^{4n} M^{2n/3}, \quad (22)$$

where  $d(n) = [C_1(n)/4C_2(n)]^{1/3}$ .

We now eliminate  $\beta$  in (20) in favor of  $M$  to obtain:

$$L_3 = \frac{1}{4} (q\dot{M} - \dot{q}M) - H_3(M), \quad (23)$$

where

$$H_3(M) = (C_2d^4 - C_1d) M^{5/3}. \quad (24)$$

We now have unconstrained Lagrangian dynamics with a conserved Hamiltonian  $H_3$ , which is the Hamiltonian (3) evaluated with the trial wave function. The set of Lagrange equations resulting from (23) are

$$\begin{aligned} \dot{M} = 0 &\implies M = \text{const.} \\ &\implies \beta = \text{const.} \end{aligned} \quad (25)$$

and

$$\dot{q} = \frac{10}{3} M^{2/3} [C_1d - C_2d^4], \quad (26)$$

as well as a conserved energy

$$E = (C_2 d^4 - C_1 d) M^{5/3} \quad (27)$$

Thus the velocity of the soliton is constant and related to the conserved energy via

$$\dot{q} = c = -\frac{10}{3}EM^{-1}. \quad (28)$$

Thus we have

$$u_v(x, t) = -d(n)M^{2/3}2^{1/4n} \left[ 2\Gamma\left(1 + \frac{1}{2n}\right) \right]^{-1/2} \times \exp\left[-d^{4n}M^{2n/3}|x - ct - x_0|^{2n}\right], \quad (29)$$

with  $c$  given by (27) and (28). To find the “best soliton” for our class of trial wave functions we need to minimize the energy of the soliton with respect to  $n$ . In Fig.1 we plot the dimensionless energy  $E(n)/M^{5/3}$ . We see that the energy is minimized for  $n = 0.877$ ; at that value we find that  $E = -0.3925M^{5/3}$ .

The exact single soliton solution of the KdV equation is given by [6]

$$u(x, t) = -\frac{c}{2}\text{sech}^2\left(\frac{1}{2}c^{1/2}(x - ct - x_0)\right), \quad (30)$$

where  $c = (3/2)^{2/3}M^{2/3} = 1.310M^{2/3}$ . Using (3), we find that the energy of the exact soliton solution is given by

$$E = -\frac{1}{5}\left(\frac{3}{2}\right)^{5/3}M^{5/3} = -0.3931M^{5/3}. \quad (31)$$

Comparing (27) and (31), we find that the energy of the true soliton is lower than that of the variational approximation (as it must be), but the error is only 0.15%. Using the proportionality of  $c$  and  $E$ , we find that the variational calculation gives for the soliton velocity at the optimal  $n$ :

$$v_0 = 1.308M^{2/3} \quad (32)$$

which is accurate to 0.2%.

In Fig. 2 we plot the best variational soliton wave function  $u_v^2$  with  $n = 0.877$  and the exact single soliton solution  $u^2$ . To show the accuracy in

detail we plot  $u_v^2(x) - u^2(x)$  in Fig. 3; the worst error (at the origin) is about 0.5%.

From these calculations we conclude that by optimizing the class of non-Gaussian variational vave functions we get an extremely accurate approximate soliton with correct velocity and energy.

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