On Pure Lattice Chern-Simons Gauge Theories

F. Berruto[a], M.C. Diamantini[b]* and P. Sodano[a]

a)*Dipartimento di Fisica and Sezione I.N.F.N., Università di Perugia, Via A. Pascoli I-06123 Perugia, Italy*

b)*Department of Theoretical Physics, Oxford University, 1 Keble Rd, Oxford UK*

Abstract

We revisit the lattice formulation of the Abelian Chern-Simons model defined on an infinite Euclidean lattice. We point out that any gauge invariant, local and parity odd Abelian quadratic form exhibits, in addition to the zero eigenvalue associated with the gauge invariance and to the physical zero mode at $\vec{p} = \vec{0}$ due to translational invariance, a set of extra zero eigenvalues inside the Brillouin zone. For the Abelian Chern-Simons theory, which is linear in the derivative, this proliferation of zero modes is reminiscent of the Nielsen-Ninomiya no-go theorem for fermions. A gauge invariant, local and parity even term such as the Maxwell action leads to the elimination of the extra zeros by opening a gap with a mechanism similar to that leading to Wilson fermions on the lattice.

arXiv:hep-th/0004203v3 22 Jul 2000

It is by now well known that in odd space-time dimensions, there is the possibility of adding a gauge invariant, topological Chern-Simons (CS) term to the gauge field action. The CS term breaks both the parity and time-reversal symmetries and, when coupled with a Maxwell or Yang-Mills term, leads to massive gauge excitations [1]. For an Abelian model in three space-time dimensions, the pure CS Lagrangian is defined as

$$\mathcal{L}_{CS} = \frac{k}{2} A_\mu \epsilon^{\mu\alpha\nu} \partial_\alpha A_\nu \quad , \tag{1}$$

where k is a dimensionless coupling constant.

The pure CS theory is a topological field theory [2]. It is exactly solvable and it is used to compute topological invariants of three manifolds, the knot invariants for links embedded in three-manifolds [2]. As a model for physical phenomena, being dominant at large distances, the CS action may be used as a low energy effective field theory for condensed matter systems such as the fractional quantum Hall effect [3] or Josephson junction arrays [4]. Of great interest is also the relationship of CS theories to conformal field theories in two dimensions [5].

While in the continuum the pure CS theory is exactly solvable, things are quite different on the lattice. As originally shown by Fröhlich and Marchetti [6], the kernel defining the CS action exhibits a set of zeroes which are not due to gauge invariance; thus, the theory is not integrable even after gauge fixing. The action (1) is of first order in the derivatives, and the appearence of extra zeros in its lattice formulation is reminescent of the "doubling" of fermions on the lattice [7, 8]. While, for fermion models, the doubling made for many years impossible to formulate chiral gauge theories on the lattice (for recent development on this subject see [9]), the extra zeroes of the CS action make only the definition of a parity odd theory on the lattice sick. In the following we revisit the lattice formulation of the Euclidean version of the Abelian CS model defined by the Lagrangian (1). Previous studies of pure CS theory on the lattice have been carried out using the Hamiltonian formalism in [10], by introducing a mixed CS action with two gauge fields with opposite parity or by means of two gauge fields living on the links of two dual lattices (thereby obtaining in both cases a parity even action) [11, 12].

In this letter we shall show that, due to the Poincaré lemma recently proved on the lattice by Lüscher [13], the non-integrability of the CS kernel is a general feature of any gauge-invariant, local and parity odd gauge theory on a lattice. Moreover, again as a consequence of the Poincaré lemma, we shall show that the only gauge invariant regularization of the CS action may be obtained by adding to it a parity even term; of course, the most physical choice is the Maxwell term. Since the addition of a Maxwell term regularizes the CS action, the presence of the extra zeros in the CS action did not cause any problem in previous investigations of the Maxwell-CS action on the lattice [6, 14, 15].

We consider an infinite Euclidean cubic lattice with lattice spacing a, which we set to unity ($a = 1$). We shall denote a lattice site by the vector $\vec{x}$ and a link between $\vec{x}$ and $\vec{x} + \hat{\mu}$ ($\mu = 0, 1, 2$) by $(\vec{x}, \hat{\mu})$. Forward and backward difference operators are given by $d_\mu f(\vec{x}) = f(\vec{x} + \hat{\mu}) - f(\vec{x}) = (S_\mu - 1)f(\vec{x})$ and $\hat{d}_\mu f(\vec{x}) = f(\vec{x}) - f(\vec{x} - \hat{\mu}) = (1 - S_\mu^{-1})f(\vec{x})$, where $S_\mu f(\vec{x}) = f(\vec{x} + \hat{\mu})$, $S_\mu^{-1} f(\vec{x}) = f(\vec{x} - \hat{\mu})$ are the forward and backward shift operators respectively. Summation by parts on the lattice interchanges the forward and backward derivatives: $\sum_{\vec{x}} f(\vec{x}) d_\mu g(\vec{x}) = - \sum_{\vec{x}} \hat{d}_\mu f(\vec{x}) g(\vec{x})$.

The lattice Fourier transformation of the gauge field A_μ is given by

$$A_\mu(\vec{x}) = \int_{\mathcal{B}} \frac{d^3 p}{(2\pi)^3} e^{-i\vec{p}\cdot\vec{x}} e^{-ip_\mu/2} A_\mu(\vec{p}) \quad . \tag{2}$$

Due to the phase factor $e^{-ip_\mu/2}$, $A_\mu(\vec{p})$ is antiperiodic if $p_\mu \longrightarrow p_\mu + 2\pi(2n+1)$, with n integer. The integration over momenta in eq.(2) is restricted to the Brillouin zone $\mathcal{B} = \{p_\mu | -\pi \le p_\mu \le \pi , \ \mu = 0, 1, 2\}$. Under parity, which on an Euclidean cubic lattice corresponds to the simultaneous inversion of all three directions, $A_\mu(\vec{x}) \longrightarrow A_\mu(-\vec{x} - \hat{\mu})$ and $A_\mu(\vec{p}) \longrightarrow -A_\mu(-\vec{p})$.

The CS action on an Euclidean lattice derived by Frölich and Marchetti [6] is:

$$S = \sum_{\vec{x}} A_\mu(\vec{x}) \tilde{K}_{\mu\nu}(\vec{x} - \vec{y}) A_\nu(\vec{y}) \quad , \tag{3}$$

where $\tilde{K}_{\mu\nu} = K_{\mu\nu} + \hat{K}_{\mu\nu}$, and

$$K_{\mu\nu}(\vec{x} - \vec{y}) = S_\mu^{\vec{y}} \, \epsilon_{\mu\alpha\nu} \, d_\alpha^{\vec{y}} \, \delta_{\vec{x},\vec{y}} \quad , \tag{4}$$

$$\hat{K}_{\mu\nu}(\vec{x} - \vec{y}) = S_\nu^{-1,\vec{y}} \, \epsilon_{\mu\alpha\nu} \, \hat{d}_\alpha^{\vec{y}} \, \delta_{\vec{x},\vec{y}} \quad . \tag{5}$$

K and $\hat{K}$ are exchanged by summation by parts.

Both K and $\hat{K}$ define a gauge invariant and parity odd kernel. In momentum space, apart from the zero due to gauge invariance, K and $\hat{K}$ have eigenvalues $\lambda(p) = \hat{\lambda}^\dagger(p) = \pm 2 e^{-i\sum_{\mu=0}^{2} p_\mu/2} \sqrt{\sum_{\mu=0}^{2} \sin^2 p_\mu/2}$ and exhibit no extra zeroes apart from the one at zero momentum, associated with translational invariance. The kernels (4) and (5), after gauge fixing, both define an integrable CS action. However, since (3) is a quadratic form in $A_\mu(\vec{x})$, the kernel should be symmetric under the simultaneous exchange of $\mu \to \nu, (\vec{x}) \to (\vec{y})$ (according to [16], we call this Bose symmetry); it is easy to see that only the linear combination $\tilde{K} = K + \hat{K}$ respects this symmetry and thus provides an acceptable definition of the lattice CS action. This has far reaching consequences on the integrability of the action (3) since, in momentum space, the operator $\tilde{K}(p) = K(p) + \hat{K}(p)$ has, apart from the zero mode associated with gauge invariance, eigenvalues given by

$$\tilde{\lambda}(p) = \pm 2 \sqrt{1 + \cos \sum_{\mu=0}^{2} p_\mu} \sqrt{3 - \sum_{\mu=0}^{2} \cos p_\mu} \quad . \tag{6}$$

One gets $\tilde{\lambda} = 0$ whenever $\cos \sum_{\mu=0}^{2} p_\mu = -1$, i.e. when $\sum_{\mu=0}^{2} p_\mu = (2n+1)\pi$, which defines planes of zeros with co-dimension 1 in the Brillouin zone: the CS action (3) is thus not integrable. The properties of K and $\hat{K}$ parallel the ones of the forward and backward derivatives, which in momentum space read $d_\mu \to e^{ip_\mu/2}\hat{p}_\mu$ and $\hat{d}_\mu \to e^{-ip_\mu/2}\hat{p}_\mu$ with $\hat{p}_\mu = 2 \sin p_\mu/2$: they do not have extra zeroes inside the Brillouin zone, but their linear combination $d + \hat{d} \to 2 \cos(p_\mu/2)\hat{p}_\mu$ has zeros at the border of the Brillouin zone $p_\mu = \pm\pi$.

The appearance of the extra zeroes is not due to the specific form of the kernel in (3). In fact, let us consider an action given by

$$S = \sum_{\vec{x},\vec{y}} A_\mu(\vec{x}) G_{\mu\nu}(\vec{x} - \vec{y}) A_\nu(\vec{y}) \quad . \tag{7}$$

2

We shall now prove that, under the assumptions that
i) (7) is local on the lattice;
ii) (7) is gauge invariant: $\hat{d}^x_\mu G_{\mu\nu}(\vec{x} - \vec{y}) = \hat{d}^y_\nu G_{\mu\nu}(\vec{x} - \vec{y}) = 0$;
iii) (7) is odd under parity;
(7) is not integrable.

The fact that S is a quadratic form implies that $G_{\mu\nu}(\vec{x}-\vec{y}) = G_{\nu\mu}(\vec{y}-\vec{x})$ (Bose symmetry). By Fourier transforming eq.(7), one gets

$$\int_{\mathcal{B}} \frac{d^3 p}{(2\pi)^3} A_\mu(-\vec{p}) \tilde{G}_{\mu\nu}(\vec{p}) A_\nu(\vec{p}) \tag{8}$$

with $\tilde{G}_{\mu\nu}(\vec{p}) = e^{ip_\mu/2} G_{\mu\nu}(\vec{p}) e^{-ip_\nu/2}$ (no sum over μ and ν). Note that, in order for $G_{\mu\nu}(\vec{p})$ to be a periodic function of the momentum $\vec{p}$, $\tilde{G}_{\mu\nu}(\vec{p})$ must be antiperiodic in $p_\mu \to p_\mu + 2\pi(2n+1)$ and $p_\nu \to p_\nu + 2\pi(2n+1)$, with n integer. Moreover, $\tilde{G}_{\mu\nu}(\vec{p})$ must be a periodic function of the component of the three-momentum different from μ and ν, since no extra phase is present in the definition of $\tilde{G}_{\mu\nu}(\vec{p})$. These properties of periodicity and antiperiodicity are crucial in the proof of the non-integrability of (7).

The kernel $\tilde{G}_{\mu\nu}(\vec{p})$ is such that

$$\tilde{G}_{\mu\nu}(\vec{p}) = \tilde{G}_{\nu\mu}(-\vec{p}) \tag{9}$$

due to the Bose symmetry. In momentum space the assumptions i)-iii) imply that $\tilde{G}_{\mu\nu}(\vec{p})$ is an analytic function (from locality) satisfying to

$$\hat{p}_\mu \tilde{G}_{\mu\nu}(\vec{p}) = 0 \tag{10}$$

from gauge invariance, and to

$$\tilde{G}_{\mu\nu}(\vec{p}) = -\tilde{G}_{\nu\mu}(\vec{p}) \tag{11}$$

from parity oddness. As a consequence of (10), using the Poincaré lemma, one may write

$$\tilde{G}_{\mu\nu}(\vec{p}) = \epsilon_{\rho\mu\nu} \hat{p}_\rho f(\vec{p}), \tag{12}$$

where $f(\vec{p})$ is an analytic function of $\vec{p}$. Since $\tilde{G}_{\mu\nu}(\vec{p})$ must be periodic in the component of the momentum different from μ and ν, one has that $f(\vec{p})$ must be antiperiodic in $p_\rho \to p_\rho + 2\pi(2n+1)$. Moreover, since all the dipendence of $\tilde{G}_{\mu\nu}(\vec{p})$ on p_μ and p_ν is carried by $f(\vec{p})$, the latter must also be antiperiodic with respect to p_μ and p_ν: in order for the kernel $G_{\mu\nu}(\vec{p})$ to be a periodic function of the momenta, $f(\vec{p})$ must satisfy to

$$f(p_0 + 2\pi(2n+1), p_1 + 2\pi(2n+1), p_2 + 2\pi(2n+1)) = -f(p_0, p_1, p_2) \quad . \tag{13}$$

Due to eqs.(11,12) one has that $f(\vec{p}) = f(-\vec{p})$. From eq.(13), one may easily check that for $p_0 = p_1 = p_2 = \pm\pi$ one gets

$$f(\pm\pi, \pm\pi, \pm\pi) = -f(\pm\pi, \pm\pi, \pm\pi) = 0 \tag{14}$$

(this is also true when any two component of the momentum are equal to zero and one is equal to $\pm\pi$). Since the spectrum of $G_{\mu\nu}(\vec{p})$ is given by $G(\vec{p}) = \pm|f(\vec{p})|\sqrt{\sum_{\mu=0}^{2} \hat{p}_\mu^2}$ (apart

3

from the zero due to gauge invariance), eq.(14) implies that the kernel $G(\vec{p})$ exhibits extra zeroes at the edges of the Brillouin zone and is thus not integrable. Note that the presence of the extra zeros is completely independent on the nature, complex or real, of the function $f(\vec{p})$. This observation is pertinent since, in Euclidean space-time, the pure CS action is purely imaginary.

Relaxing the assumption iii) one may study the general form of a gauge invariant local action in three dimensions. With the help of the Poincaré lemma [13] it is easy to show that the kernel $\tilde{G}_{\mu\nu}(\vec{p})$ can be divided into the sum of parity even and parity odd terms. Since, due to locality, $\tilde{G}_{\mu\nu}(\vec{p})$ is an analytic function of $\vec{p}$, it may be expanded in Taylor series: all the terms having even power of the momenta are parity even, while the terms with odd power of the momenta are parity odd. The terms with the lowest number of derivatives in this expansion are the CS term defined in [6] and the Maxwell term, whose kernel on the lattice is:

$$M_{\mu\nu} = -\Box\delta_{\mu\nu} + d_\mu\hat{d}_\nu = K_{\mu\rho}\hat{K}_{\rho\nu}, \tag{15}$$

where $\Box = \sum_{\mu=0}^{2} d_\mu\hat{d}_\mu$ is the Laplacean in three dimensions. Since all the parity odd terms fullfill the assumptions of the theorem they generate extra zeroes in the spectrum. The only gauge invariant way to regularize the CS action is then the addition of a parity even term such as the Maxwell term.

For the Maxwell-CS theory on the lattice the kernel $\Gamma_{\mu\nu}$ may be written as

$$\Gamma_{\mu\nu} = \frac{1}{4e^2}M_{\mu\nu} + ikG_{\mu\nu}. \tag{16}$$

In (16) k is dimensionless and e^2 has the dimension of a mass; the Maxwell term is an irrelevant operator and the CS action dominates in the infrared region. The Fourier transform of $\Gamma_{\mu\nu}$, apart from a zero mode due to gauge invariance, has eigenvalues given by

$$\lambda_{MCS}(\vec{p}) = \frac{1}{2e^2}\sum_{\mu=0}^{2}(1 - \cos p_\mu) + ikG(\vec{p}) \quad, \tag{17}$$

and, as it stands, it is free from extra zeroes in the Brillouin zone since the first term in (17), which is the Fourier transform of the Maxwell kernel, is zero only at zero momentum, and at the corners of the Brillouin zone, $p_\mu = \pm\pi$, $\mu = 0, 1, 2$, takes the value $\lambda_{MCS}(\vec{p}) = 3/e^2$.

Since the CS action is purely immaginary, the addition of the Maxwell term is used also in the continuum theory to provide a proper definition of the functional integral in the partition function of the pure CS theory. The CS limit is reached also there by taking the limit $e^2 \longrightarrow \infty$ after Gaussian integration.

The regularization of the extra zeros in the CS action by adding a Maxwell term and thereby opening a gap in the fermion spectrum is similar to the mechanism of the Wilson fermion where a gap is opened and the energy does not have secondary minima at the non-zero corners of the Brillouin zone. As in the case of the Wilson fermions [17], the regularization is done by means of an irrelevant operator and the continuum limit $a \longrightarrow 0$ is not changed by this addition. Moreover, as the Wilson action explicitly breaks chiral symmetry, the action obtained after the addition of the Maxwell term is not anymore defined under parity.

4

A key result of our analysis is a no-go theorem in the lattice regularization of the pure CS theory, if one requires locality, gauge invariance and parity oddness on the lattice. Clearly all the arguments needed for the proof rely on the definition of the parity transformation for the gauge field, i.e. on $A_\mu(\vec{x}) \longrightarrow A_\mu(-\vec{x}-\hat{\mu})$. One possible way out is to define a new parity transformation under which the CS action is still odd but the kernel is integrable. The new definition of the parity transformation should then play a role analogous to the Ginsparg-Wilson relation [18] in the definition of chiral gauge theories on the lattice. When the Ginsparg-Wilson relation holds, the lattice fermion action has an exact chiral symmetry [19] and the no-go theorem of Nielsen and Ninomiya [20] is avoided. It is pertinent to point out that, while for chiral gauge theories the problem of the non-perturbative regularization of the theory in a way preserving the symmetry is related to the presence, at the quantum level, of anomalies, for the CS action the theory in the continuum is well defined and solvable. This hints to the fact that the non-integrability on the lattice of the CS action is only a lattice artifact, due mainly to the existence of two derivatives giving rise to the two kernels K and $\hat{K}$.

References

[1] R. Jackiw and S. Templeton, Phys. Rev. **D23** (1981) 2291; J. Schonfeld, Nucl. Phys. **B185** (1981) 157; S. Deser, R. Jackiw and S. Templeton, Phys. Rev. Lett. **48** (1982) 957, Ann. Phys. (N.Y.) 140 (1982) 372.

[2] E. Witten, Comm. Math. Phys. **121** (1989) 351, Int. Jour. Mod. Phys. **A6** (1991) 2775.

[3] For a review see: A Zee, Prog. Theor. Phys. Supp. **107** (1992) 77.

[4] M.C. Diamantini P. Sodano and C.A. Trugenberger, Nucl. Phys. B 448, (1995) 505, Nucl. Phys. B 474 (1996) 641

[5] E. Witten, Commun. Math. Phys. **144** (1992) 189, M. Blau and G. Thompson, Nucl. Phys. **B408** (1993) 345; M. Bos and V. P. Nair, Phys. Lett. **B223** (1989) 61.

[6] J. Frölich and P.-A. Marchetti, Comm. Math. Phys. **121** (1989) 177.

[7] J. Kogut and L. Susskind, Phys. Rev. **D11** (1975) 395; L. Susskind, Phys. Rev. **D16** (1977) 3031.

[8] K. G. Wilson, "New Phenomena in Subnuclear Physics" (Erice, 1975), ed. A. Zichichi (New York, Plenum, 1975).

[9] F. Niedermayer, Nucl. Phys. **B** (Proc. Suppl.), 73 (1999) 105; H. Neuberger, *Chiral fermions on the lattice*, hep-lat/9909042; M. Lüscher, *Chiral gauge theories on the lattice with exact gauge invariance*, hep-lat/9909150.

[10] D. Eliezer and G. W. Semenoff, Ann. of Phys. **217** (1992) 66; D. Eliezer and G. W. Semenoff, Phys. Lett. **B286** (1992) 118.

[11] R. Kantor and L. Susskind, Nucl. Phys. **B366** (1993) 533;D. Eliezer and G. W. Semenoff, Phys. Lett. **B266**, (1991) 375.

[12] D. H. Adams, Phys. Rev. Lett. **78** (1997) 4155.

[13] M. Lüscher, Nucl. Phys. **B538** (1999) 515.

[14] M. Lüscher, Nucl. Phys. **B326** (1989) 557; V.F. Müller, Z. Phys. C **47** (1990) 301.

[15] M.C. Diamantini, P. Sodano and C.A. Trugenberger, Phys. Rev. Lett. **71** (1993) 1969, Phys. Rev. Lett. **75** (1995) 3517.

[16] L. H. Karsten and J. Smit, Nucl. Phys. **B183** (19981) 103.

[17] J. Kogut and K.G. Wilson, Phys. Rev. **C12** (1974) 75.

[18] P.H. Ginsparg and K.G. Wilson, Phys. Rev **D25** (1982) 2649.

[19] M. Lüscher, Phys. Lett. **B428** (1998) 342.

[20] H. B. Nielsen and M. Ninomiya, Nucl. Phys. **B185** (1981) 20, Nucl. Phys. **B193** (1981) 173, Phys. Lett. **B105** 219 (1981), Phys. Lett. **B130** 389 (1983).