Physica 137A (1986) 282–294
North-Holland, Amsterdam

NONLINEAR σ-MODELS WITH NONCOMPACT SYMMETRY GROUP IN THE THEORY OF A NEARLY IDEAL BOSE GAS

O.K. PASHAEV and S.A. SERGEENKOV
Joint Institute for Nuclear Research, Dubna, USSR

Received 5 November 1985

A classical version of the Heisenberg spin model on the noncompact SU(1, 1)/U(1) manifold is constructed which is gauge equivalent to the NLSE. It is found that the gauge transformations generated by the Jost solutions to the NLSE linear problem allows one to obtain solutions of the appropriate σ-model. Spin-wave and soliton solutions and related energy, momentum and magnetization integrals are found. The spin-waves describe a precession motion on the pseudosphere $S^{1,1}$ with the Bogolubov frequency, and the soliton solution describes a deviation from the precession motion plane. The soliton excitation spectrum when condensate density vanishes is reduced to that of the O(3) Heisenberg ferromagnet. In the case of unlimited length of the magnetization vector the first one gives rise to the hole excitation spectrum of an antiferromagnet, and magnetizations related to the upper and lower sheets of the hyperboloid compensate each other.

The investigation has been performed at the Laboratory of Computing Techniques and Automation, JINR.

1. Introduction

The use of sigma models with field values on noncompact manifolds has attracted much interest recently[1]). Such models arise in field theories of gravity[2]) and extended supergravity[3]), in the Anderson localization problem[4]) and in string models[5]). It is well known that the classical Heisenberg ferromagnet model in the continuum limit can be formulated in the form of a nonlinear O(3) σ-model[6]). The question arises: if the Heisenberg model on noncompact manifolds is considered, to which physical model may it be related? One can study this problem in one dimension, when the isotropic Heisenberg model becomes the integrable system[7]). As is well known, the classical integrable systems may be mapped into the quantum ones which are exactly solvable via the quantum spectral transform. Moreover, there exists a gauge equivalence between some completely integrable equations. In the framework of the gauge equivalence the connection between the O(3) Heisenberg ferromagnet model

0378-4371/86/$03.50

(described by the isotropic Landau–Lifshitz equation) and the attractive nonlinear Schrödinger equation (NLSE) has been established[8,9]. Furthermore, the gauge equivalence between the easy-axis type anisotropic Landau–Lifshitz equation and the isotropic Landau–Lifshitz equation[10] as well as the attractive type NLSE has been found [11–13].

It is important to notice that the classical spin S manifolds for the isotropic model essentially depend on the sign of the anisotropy term in the corresponding anisotropic model[10]. For example, in the case of the easy-plane Heisenberg model with $S^2 = 1$ and hence $S \in SU(2)/U(1)$, the gauge-equivalent isotropic Heisenberg model is defined on a noncompact manifold belonging to $SL(2, C)/U(1)$ [10], and the corresponding NLSE describes a mixture of attractive and repulsive Bose gases[14,15]. Moreover, the attractive gas configurations are determined by the repulsive ones and the corresponding constraint on the appropriate field variables leads to an exotic NLSE modification[16].

On the other hand a reasonable question is: what Heisenberg model is gauge-equivalent to the repulsive NLSE? As has been shown[17,18] the corresponding model is described by the isotropic Landau–Lifshitz equation with classical spin which belongs to the noncompact $SU(1,1)/U(1)$ manifold with the constant negative curvature (the Lobachevsky plane).

In the present paper using the gauge transformations generated by the Jost solutions of the NLSE linear problem we obtain solutions of the $O(2,1)$ Heisenberg model. Two type solutions are found: spin-waves and solitons. The spin-waves describe a precession motion on the pseudosphere $S^{1,1} = SU(1,1)/U(1)$ about the z axis with the Bogolubov frequency and the soliton solution describes a deviation from the precession motion plane. The related energy, momentum, magnetization integrals and corresponding spectrum will be found. It is shown that the spin-wave spectrum is determined by the Bogolubov dispersion (as in the theory of a weakly nonideal Bose gas) and the soliton spectrum has two remarkable limits. In the first one, when the pseudospin vector S evolves in a neighbourhood of the lowest point of the upper sheet of $S^{1,1}$ (which corresponds to the vanishing condensate density in the related Bose gas) the soliton excitation spectrum is reduced to that of the $O(3)$ Heisenberg ferromagnet and coincides with the exact one[6]. In the case of unlimited length of the pseudospin vector S this spectrum gives rise to the hole excitation spectrum of antiferromagnet[19]. Moreover, in the latter case the magnetizations related to the upper and lower sheets of the hyperboloid compensate each other. These facts* allow to conclude that it may well be that the noncompact Heisenberg model has a more straightforward relation to theory of magnetism.

* As is well known the two sublattice ferrimagnets establish a continuous transition between ferro- and antiferromagnets.

Besides, as it has been shown in the recent papers[20]) the classical dynamics of linearized two sublattice antiferro- and ferrimagnets is reproduced as a harmonic motion on the Lobachevsky plane (a noncompact manifold with the negative constant curvature).

2. Formulation of the problem

The noncompact Heisenberg model is governed by the isotropic Landau–Lifshitz equation

$$S_t = \frac{1}{2\mathrm{i}}\,[S, S_{xx}]\,, \tag{1}$$

where

$$S(x,t) = \begin{pmatrix} S^z & \mathrm{i}S^- \\ \mathrm{i}S^+ & -S^z \end{pmatrix} \in \mathrm{su}(1,1)\,, \quad S^\pm = S^x \pm \mathrm{i}S^y\,,$$

and

$$\det S = -1\,, \quad S^2 = I\,. \tag{2}$$

It follows from eq. (2) that the magnetization or pseudospin vector $\boldsymbol{S} = (S^x, S^y, S^z)$ satisfies the condition

$$(S^z)^2 - (S^x)^2 - (S^y)^2 = 1 \tag{3}$$

and does not have the fixed length $\boldsymbol{S}^2 = 2S_z^2 - 1$. Eq. (3) describes two sheet hyperboloid (pseudosphere $\mathrm{S}^{1,1}$). Let us expand the matrix $S(x,t)$ into the basis of the su(1, 1) Lie algebra

$$S(x,t) = \sum_{\alpha=1}^{3} S^\alpha(x,t)\tau_\alpha\,, \tag{4}$$

where

$$\tau_\alpha \tau_\beta = g_{\alpha\beta} + \mathrm{i}f_{\alpha\beta\gamma}\tau_\gamma \quad (\alpha, \beta, \gamma = 1, 2, 3)\,,$$

$g_{\alpha\beta} = \mathrm{diag}(-1, -1, 1)$ is the Killing metrics, and $f_{\alpha\beta\gamma}$ are the structure constants of the su(1, 1) algebra

$$\mathrm{Tr}(\tau_\alpha \tau_\beta) = 2g_{\alpha\beta}\,, \quad [\tau_\alpha, \tau_\beta] = 2\mathrm{i}f_{\alpha\beta\gamma}\tau_\gamma\,. \tag{5}$$

In terms of the pseudospin field components the equations of motion take the form

$$\dot{S}^\alpha = \sum_{\beta,\gamma} f^{\alpha\beta\gamma} S_\beta S_{\gamma xx} \,. \tag{6}$$

These equations may be presented in the Hamiltonian form

$$\dot{S}^\alpha(x, t) = \{H, S^\alpha(x, t)\} \,,$$

where the Poisson bracket algebra is

$$\{S^\alpha(x), S^\beta(y)\} = -f^{\alpha\beta\gamma} S^\gamma(x) \delta(x - y) \,. \tag{7}$$

The corresponding Hamiltonian function is

$$H = \tfrac{1}{2} \int_{\infty}^{\infty} \mathrm{d}x\, S^\alpha_x g_{\alpha\beta} S^\beta_x = \tfrac{1}{4} \operatorname{Tr} \int_{-\infty}^{\infty} \mathrm{d}x (S_x)^2 \,. \tag{8}$$

After parametrizing S by two angle variables,

$$\begin{aligned} S^x(x, t) &= \operatorname{sh} \theta(x, t) \cos \varphi(x, t) \,, \\ S^y(x, t) &= \operatorname{sh} \theta(x, t) \sin \varphi(x, t) \,, \\ S^z(x, t) &= \operatorname{ch} \theta(x, t) \,, \end{aligned} \tag{9}$$

the Hamiltonain function (8) becomes the energy of the free static scalar field $l(x)$ defined in the Lobachevsky space with the metrics $-\mathrm{d}l^2 = \mathrm{d}\theta^2 + \operatorname{sh}^2 \theta \, \mathrm{d}\varphi^2$

$$H = \tfrac{1}{2} \int_{-\infty}^{\infty} \mathrm{d}x \left(\frac{\mathrm{d}l}{\mathrm{d}x} \right)^2 = -\tfrac{1}{2} \int_{-\infty}^{\infty} \mathrm{d}x \left[\left(\frac{\mathrm{d}\theta}{\mathrm{d}x} \right)^2 + \operatorname{sh}^2 \theta \left(\frac{\mathrm{d}\varphi}{\mathrm{d}x} \right)^2 \right] .$$

The model interpretation in the nonideal Bose gas picture is most obvious in terms of the pseudospherical projection, which is introduced through the relations

$$S^+ = \frac{2\sqrt{\rho}\xi}{\rho - |\xi|^2} \,, \quad S^z = \frac{\rho + |\xi|^2}{\rho - |\xi|^2} \,, \tag{10}$$

where $\xi(x, t)$ is the complex field defined on the plane $(\operatorname{Re} \xi, \operatorname{Im} \xi)$, and ρ is constant ($\rho > 0$). It follows from eq. (10) that the upper sheet of the hyperboloid $S^z = +\sqrt{1 + S^+ S^-} > 0$ is mapped to the inner part of the circle with the radius

$\sqrt{\rho}$: $D_+ = \{\xi(x, t)\colon |\xi|^2 < \rho\}$, and the lower sheet $S^z = -\sqrt{1 + S^+S^-} < 0$ is mapped to the outer part of the same circle: $D_- = \{\xi(x, t)\colon |\xi|^2 > \rho\}$. Moreover, the lowest point $S_z = +1$ of the upper sheet is mapped to the centre of the circle $\xi = 0$, and the highest point $S^z = -1$ of the lower sheet goes to infinity $|\xi| \to \infty$. When the magnetization length grows unlimitedly and $S^z \to \pm\infty$, the related points on the ξ plane approach to the circle circumference from the inner and outer sides $|\xi| \to \sqrt{\rho} \mp 0$ correspondingly. Therefore in the points of the circumference (into which the "light cone" is mapped) we can define in addition the magnetization of the upper and lower sheets to compensate them reciprocally. The Poisson bracket between the fields $\xi(x)$ and $\xi^*(x)$ is

$$\{\xi(x), \xi^*(y)\} = (\rho - |\xi|^2)^2 \delta(x - y)\,. \tag{11}$$

It is easy to see that when $|\xi|^2 \ll \rho$ (in the neighbourhood of the point $\xi = 0$), the Poisson bracket (11) takes the canonical form as well as in the compact O(3) Heisenberg model. However in contrast to the latter case owing to the presence of the minus sign in eq. (11) there arises one more nontrivial possibility. When $|\xi|^2 \approx \rho$ the bracket (11) vanishes. On the quantum level this fact should indicate to the presence of the macroscopic (classical) state which is similar to the Bose condensate in the Bogolubov theory of superfluidity.

The field equations (6) in terms of $\xi(x, t)$ take the form of a modified NLSE

$$\mathrm{i}\xi_t + \xi_{xx} + 2\,\frac{\xi^*(\xi_x)^2}{\rho - |\xi|^2} = 0 \tag{12}$$

and the Hamiltonian function (8) takes the form

$$H = -2\rho \int_{-\infty}^{\infty} \mathrm{d}x\, \frac{|\xi_x|^2}{(\rho - |\xi|^2)^2} = -2\rho \int_{-\infty}^{\infty} \mathrm{d}x\; g(\xi, \xi^*)|\xi_x|^2\,, \tag{13}$$

which looks like the Ernst Lagrangian function in the theory of axially symmetric gravity. Here $g(\xi, \xi^*) = (\rho - |\xi|^2)^{-2}$ is the metrics of the Poincare model in the region $D_+(|\xi|^2 < \rho)$. It is interesting to note that Hamiltonian (13) describes the energy of the free complex scalar field $\xi(x)$ in the curved "charge space" with the constant negative curvature.

3. The gauge equivalence and solitons

There is a gauge equivalence between the repulsive NLSE

$$\mathrm{i}\psi_t + \psi_{xx} - 2(|\psi|^2 - \rho)\psi = 0 \tag{14}$$

and the O(2, 1) Landau–Lifshitz equation (1) [17,18]). The linear problem (the Lax pair) corresponding to eq. (14) is [22,23])

$$\Phi_{1_x} = U_1(x, t; \lambda)\Phi_1 , \quad \Phi_{1_t} = V_1(x, t; \lambda)\Phi_1 , \tag{15}$$

where

$$U_1(x, t; \lambda) = -\mathrm{i}\lambda\sigma_3 + \begin{pmatrix} 0 & \mathrm{i}\psi^* \\ -\mathrm{i}\psi & 0 \end{pmatrix} ,$$

$$V_1(x, t; \lambda) = \begin{pmatrix} 2\mathrm{i}\lambda^2 + \mathrm{i}(|\psi|^2 - \rho) & -2\mathrm{i}\lambda\psi^* + \psi_x^* \\ 2\mathrm{i}\lambda\psi + \psi_x & -2\mathrm{i}\lambda^2 - \mathrm{i}(|\psi|^2 - \rho) \end{pmatrix} .$$

The integrability conditions for the system (15),

$$U_{1_t} - V_{1_x} + [U_1, V_1] = 0 ,$$

leads to the NLSE (14). Let us consider the gauge transformation to new variables,

$$\begin{aligned} U_2 &= g^{-1}U_1 g - g^{-1}g_x , \\ V_2 &= g^{-1}V_1 g - g^{-1}g_t , \\ \Phi_2 &= g^{-1}\Phi_1 , \end{aligned} \tag{16}$$

which satisfy the following linear problem:

$$\Phi_{2_x} = U_2(x, t; \lambda)\Phi_2 , \quad \Phi_{2_t} = V_2(x, t; \lambda)\Phi_2 . \tag{17}$$

We choose as the gauge transformation function $g(x, t; \lambda_0)$ the normalized Jost solutions of the linear problem (15) with the fixed value λ_0 of the spectral parameter λ:

$$g(x, t; \lambda_0) \equiv \Phi_1(x, t; \lambda = \lambda_0) . \tag{18}$$

Then the matrix operators in eq. (16) becomes

$$\begin{aligned} U_2 &= -\mathrm{i}(\lambda - \lambda_0)S , \\ V_2 &= 2\mathrm{i}(\lambda^2 - \lambda_0^2)S - (\lambda - \lambda_0)SS_x , \end{aligned} \tag{19}$$

where

$$S(x, t) = g^{-1}(x, t)\sigma_3 g(x, t) \tag{20}$$

and the integrability conditions for eqs. (17) give the Landau–Lifshitz equation

$$S_t = \frac{1}{2\mathrm{i}} [S, S_{xx}] - 4\lambda_0 S_x . \tag{21}$$

By the coordinate transformations

$$t' = t , \quad x' = x - 4\lambda_0 t , \tag{22}$$

eq. (21) may be reduced to the form of the initial eq. (1) (with the appropriate changes of the boundary conditions). That is why we shall investigate eq. (1) below.

The most important case of the repulsive Bose gas problem corresponds to the finite density gas $\rho = \lim_{N\to\infty, L\to\infty} N/L$ (the thermodynamical limit). On the classical level this leads to setting of the nonvanishing boundary conditions for the field variables $\psi(x, t)$:

$$\begin{cases} \psi(x, t) \to \psi_\pm , \\ \psi_x(x, t) \to 0 , \end{cases} \quad x \to \pm\infty ,$$

where $|\psi_+|^2 = |\psi_-|^2 = \rho$ is the condensate density[15,22]). Using the Jost solutions of the linear spectral problem (15) [23]) in the asymptotic form one gets nondiagonal boundary conditions for $S(x, t)$,

$$S(x, t) \xrightarrow[x\to\pm\infty]{} S_\pm(x, t) = \frac{1}{k} \begin{pmatrix} \sqrt{k^2 + 4\rho} & 2\psi_\pm\, \mathrm{e}^{\mathrm{i}\tilde{\theta}(x,t)} \\ -2\psi_\pm^*\, \mathrm{e}^{-\mathrm{i}\tilde{\theta}(x,t)} & -\sqrt{k^2 + 4\rho} \end{pmatrix} , \tag{23}$$

where $k = 2\sqrt{\lambda_0^2 - \rho}$ is the wave number, λ_0 is the normalization point of the continuous spectrum of problem (15) ($|\lambda_0| \geqslant \sqrt{\rho}$) and $\tilde{\theta}(x, t) = k(x + vt)$, $v = \sqrt{k^2 + 4\rho}$. Solutions (23) describe a precession motion of the pseudospin vector $\boldsymbol{S}$ around the z axis with the frequency $\omega = k\sqrt{k^2 + 4\rho}$ which corresponds to the Bogolubov excitation spectrum in a weakly nonideal Bose gas of finite density ρ. For fixed value of S_z the evaluation is connected with the appearance of a phase α so that $\psi_- = \psi_+ \mathrm{e}^{\mathrm{i}\alpha}$. The precessional motion of the pseudospin vector $\boldsymbol{S}$ in space describes a spin wave propagating with the velocity v. Its amplitude depends on the wave number k. Here it is important to emphasize two circumstances. First of all, the setting of the boundary conditions (23) is possible only in the case of the "reflectionless" scattering matrix of the auxiliary linear problem (15) related to NLSE (14). So in what follows we shall use the Jost solutions describing only the scattering of the plane waves with the "reflectionless" potentials (i.e., the NLSE solitons). Then, the setting of the nonvanishing boundary conditions in the form of eq. (23) leads to the classical solutions of eq. (1) with infinite energy (as in the case of the plane waves). And only when $k \gg \sqrt{\rho}$, $S_\pm \to \sigma_3$ (the vanishing of the condensate), the energy of solutions may be finite. A similar situation takes place for the repulsive NLSE where the kink energy is divergent. After the substraction of the infinite condensate energy, the cancellation of divergencies leads to the finite result.

Using the gauge equivalence it is easy to show[14]) that any solution of eq. (1) may be transformed to the solution of eq. (14) through the relations

$$\psi(x, t) = \tfrac{1}{2}(\theta_x + \mathrm{i}\,\mathrm{sh}\,\theta \cdot \varphi_x)\,\mathrm{e}^{\mathrm{i}\tilde{\alpha}(x,t)}\,, \tag{24}$$

where

$$\tilde{\alpha}_t = \mathrm{ch}\,\theta \cdot \varphi_t - \tfrac{1}{2}(\theta_x^2 + \mathrm{sh}^2\,\theta \cdot \varphi_x^2 - 4\rho)\,, \quad \tilde{\alpha}_x = \mathrm{ch}\,\theta \cdot \varphi_x$$

(see eq. (9)). From eq. (24) it is easy to see that the problem of finding of NLSE solutions from the known solutions of the Landau–Lifshitz equation (1) is rather simple, while the inverse problem is more difficult. That is why we shall use another method corresponding in fact to solutions of this inverse problem. Namely, if we know the Jost solutions for the linear equations (15) which are related to NLSE (14) we may construct the corresponding solutions of eq. (1) via the relations (20).

To construct a one-soliton solution of the noncompact O(2, 1) Landau–Lifshitz equation (1) we use the Jost solutions of eq. (15) in the presence of one soliton[23]):

$$\Phi(x, t; \lambda, \lambda_0) = \begin{pmatrix} a & b \\ b^* & a^* \end{pmatrix}, \tag{25}$$

where

$$a(x, t; \lambda, \lambda_0) = \mathrm{e}^{-\mathrm{i}\zeta_0(x-2\lambda_0 t)}\left(\psi_+^* - \frac{\nu\psi_+^*[\rho + (\lambda - \mathrm{i}\nu)(\lambda_0 - \zeta_0)]}{\rho(\nu + \mathrm{i}\zeta_0)(1 + \mathrm{e}^{2z})}\right),$$

$$b(x, t; \lambda, \lambda_0) = \mathrm{e}^{\mathrm{i}\zeta_0(x-2\lambda_0 t)}\left((\lambda_0 - \zeta_0) - \frac{\nu[(\lambda_0 - \zeta_0) + (\lambda - \mathrm{i}\nu)]}{(\nu - \mathrm{i}\zeta_0)(1 + \mathrm{e}^{2z})}\right),$$

$$z = \nu(x - 2(\lambda - 2\lambda_0)t + x_0)\,.$$

Here, λ is a spectral parameter connected with the velocity and amplitude of the NLSE kink solution, and

$$\nu = \sqrt{\rho - \lambda^2} \quad (|\lambda| \leq \sqrt{\rho}) , \qquad \zeta_0 = \sqrt{\lambda_0^2 - \rho} \quad (|\lambda_0| \geq \sqrt{\rho}) .$$

Substituting eq. (25) in eq. (20) we find the one-soliton solution of eq. (1) in the form

$$S^z = \frac{\lambda_0}{\zeta_0} - \frac{\nu^2}{2\zeta_0(\lambda_0 - \lambda)} \operatorname{sech}^2 z ,$$
$$S^+ = \mathrm{i} \frac{\psi_+^*(\rho + \lambda\lambda_0 + \mathrm{i}\nu\zeta_0)}{2\rho\zeta_0(\nu + \mathrm{i}\zeta_0)^2} [(\lambda - \lambda_0)^2 + (\nu \operatorname{th} z + \mathrm{i}\zeta_0)^2] \, \mathrm{e}^{-2\mathrm{i}\zeta_0(x + 2\lambda_0 t)} . \tag{26}$$

It is easy to check that this solution satisfies the boundary conditions (23) if

$$\mathrm{e}^{\mathrm{i}\alpha} = \frac{\rho - \lambda\lambda_0 + \mathrm{i}\nu\zeta_0}{\rho - \lambda\lambda_0 - \mathrm{i}\nu\zeta_0} .$$

4. Integrals of motion

The equation (1) is integrable and hence it has an infinite set of integrals of motion. We are interested here only in the first three of them: the magnetization M_z along the z axis, momentum Π and energy E which are written as follows:

$$M_z = \int_{-\infty}^{\infty} \mathrm{d}x (S^z(x, t) - S_0^z) ,$$
$$\Pi = \int_{-\infty}^{\infty} \mathrm{d}x (\pi(x, t) - \pi_0) , \tag{27}$$
$$E = \int_{-\infty}^{\infty} \mathrm{d}x (\mathcal{H}(x, t) - \mathcal{H}_0) ,$$

where

$$\pi(x, t) = \frac{\mathrm{i}}{2} \frac{S_x^+ S^- - S_x^- S^+}{1 + S^z} ,$$
$$\mathcal{H}(x, t) = \tfrac{1}{2}[(S_x^z)^2 - S_x^+ S_x^-] ,$$
$$A_0 = \lim_{|x| \to \infty} A(x, t) , \quad A(x, t) = (S^z, \pi, \mathcal{H}) .$$

The variables S_0^z, π_0 and $\mathcal{H}_0$ describe the "classical vacuum state" related to the condensate state. The dynamical variables (27) for the one-soliton solution (26) are

$$\begin{aligned} M_z &= \frac{\nu}{\zeta_0(\lambda_0 - \lambda)}\,, \\ \Pi &= 4 \arcsin \frac{\nu}{\sqrt{2(\lambda_0 - \lambda)(\lambda_0 + \zeta_0)}}\,, \\ E &= 4\nu = 4\sqrt{\rho - \lambda^2}\,, \end{aligned} \tag{28}$$

with

$$S_0^z = \frac{\lambda_0}{\zeta_0}\,, \quad \pi_0 = -2(\lambda_0 - \zeta_0)\,, \quad \mathcal{H}_0 = -2\rho\,. \tag{29}$$

It is more convenient to express eqs. (28) and (29) in terms of the wave number k which defines projections of the magnetization vector $\boldsymbol{M}$ of the spin wave (23). As a result we have

$$\begin{aligned} M_z(k, \lambda) &= \frac{4\sqrt{\rho - \lambda^2}}{k(\sqrt{k^2 + 4\rho} - 2\lambda)}\,, \\ \Pi(k, \lambda) &= 4 \arcsin \left[\frac{2(\rho - \lambda^2)}{(\sqrt{k^2 + 4\rho} - 2\lambda)(\sqrt{k^2 + 4\rho} + k)} \right]^{1/2}, \\ E(k, \lambda) &= 4\sqrt{\rho - \lambda^2}\,, \end{aligned} \tag{30}$$

and, respectively,

$$S_0^z = \frac{\sqrt{k^2 + 4\rho}}{k}\,, \quad \pi_0 = -(\sqrt{k^2 + 4\rho} - k)\,, \quad \mathcal{H}_0 = -2\rho\,. \tag{31}$$

The solution (23) describes the spin wave with the Bogolubov dispersion law:

$$\omega(k) = k\sqrt{k^2 + 4\rho}\,. \tag{32}$$

When $k \gg \sqrt{\rho}$ the magnetization density S_0^z tends to unity, i.e. the pseudospin vector is near the minimum of the upper sheet of hyperboloid (the condensate is absent). The spin wave dispersion law in this case is $\omega(k) \simeq k^2$. When $k \ll \sqrt{\rho}(k/\sqrt{\rho} \to 0)$, the density S_0^z tends to infinity as $(2\sqrt{\rho}/k)_{k\to 0}$, i.e. behaves like an average particle number of the Bose condensate[24])

$$\langle a_k^+ a_k \rangle \xrightarrow[k\to 0]{} \frac{\sqrt{\rho}}{k} \to \infty .$$

Here the spin wave dispersion law becomes $\omega(k) \simeq 2\sqrt{\rho}k$ and describes collective motion of particles with the group velocity $v(|v| \geqslant |v_s|)$, where the sound velocity is

$$v_s \equiv \left(\frac{d\omega}{dk}\right)_{k=0} = 2\sqrt{\rho} .$$

It should be noted that in contrast to the nonlinear spin wave with a finite amplitude (in the compact O(3) Heisenberg model[25,26])),

$$S^+(x, t) = \sqrt{1-(S^z)^2}\, e^{i(kx-\omega t)} , \quad \omega = S^z k^2 ,$$

which is unstable and decays into some number of solitons, in our case the similar wave (23) becomes stable. Another peculiarity of the spin wave (23) lies in the fact that its energy density (31) coincides with NLSE condensate density and doesn't depend on k (due to the divergence cancellations as in U(1, 1) NLSE [18])). So the "vacuum" solution in our system is infinitely degenerate in the wave number k. This is a consequence of the noncompactness of the pseudospin manifold and the complicated vacuum structure of the corresponding quantum system.

5. Conclusion

Let us come back to the soliton solution (26) and analyse its excitation spectrum. Excluding from eq. (30) parameter λ one gets

$$E \cdot M_z = 8\left(1 + \sqrt{1 + \frac{4\rho}{k^2}}\right) \sin^2 \frac{\Pi}{4} . \tag{33}$$

Here parameter k numerates the "vacuum" states (spin waves) for which the excitation spectrum is constructed. If the condensate density in the system is rather small, i.e. $k^2 \gg \rho$, then the soliton dispersion law near the minimum of the upper sheet of the hyperboloid is

$$E(\Pi) = \frac{16}{M_z} \sin^2 \frac{\Pi}{4} . \tag{34}$$

This result coincides with one obtained in the framework of the O(3) Heisenberg model[25]) and with the exact result for the Bethe spin complex[6]). In this case M_z characterizes the quantum number of solitons and is proportional to the particle effective mass (corresponding to the soliton for small Π):

$$E(\Pi) \simeq \frac{\Pi^2}{M_z} .$$

For large density $\rho \gg k^2$ the precession frequency tends to zero and $M_z \to \infty$. In this case eq. (33) does not permit the limiting transition $k \to 0$ because M_z is no longer a dynamical variable (the precession frequency vanishes). Hence, excluding the parameter λ from the second and third equations of eqs. (30) we get the dispersion law of the soliton (26) in this case

$$E(\Pi) = 4\sqrt{\rho} \sin \frac{\Pi}{2} , \tag{35}$$

where $0 \leq \Pi \leq 2\pi$.

This dependence E on Π (up to a coefficient) coincides with the exact dispersion law for the antiferromagnon[19]) and for small Π gives*

$$E(\Pi) \simeq 2\sqrt{\rho}\Pi .$$

So, similarly to the Bogolubov dispersion law which leads in two limiting cases to quadratic and linear behaviour, the dispersion law of our soliton solution (26) leads to ferromagnet and antiferromagnet types, respectively. As is well-known, on the level of linearized theory, the ferrimagnets have a similar behaviour.

The exact coincidence of the spectra, the gauge equivalence with the repulsive Bose gas model, the compensation of the upper sheet and lower sheet magnetization when $k \to 0$ indicate on a possible connection of our model with the antiferromagnets model. It is known[27]) that the antiferromagnet phase in the XXZ model corresponds to the repulsive lattice Bose gas, while its behaviour near the critical temperature can be described in the framework of a weakly imperfect Bose gas[28]). Moreover, as is pointed out in recent papers[20]), the noncompact group $\Pi_k \otimes \mathrm{SU}(1,1)_k$ plays the role of a dynamical group for the linearized two-sublattice antiferromagnet, and the corresponding classical dynamics of the model is described by harmonic motions in the Lobachevsky plane.

It is well known[29]) that the isotropic Heisenberg model of antiferromagnet is completely integrable. Hence, the mysterious coincidence of the spectra for intergable models (as, for example, in the case of the sine-Gordon and Thirring models) can indicate to a more intimate connection between them.

* It should be noted that the two-sublattice Landau–Lifshitz model of the antiferromagnet gives linear dispersion law for the spin wave solution[6]).

Acknowledgements

The authors are indebted to Prof. V.G. Makhankov for the discussion of the problem and his interest to the work and to I. Gochev and A.M. Kosevich for a number of useful and critical remarks.

References

1) R. Geroch, J. Math. Phys. **12** (1971) 918.
E. D'Hoker, D. Freedman and R. Jackiw, Preprint MIT, CTP 1072 (1983).
J.W. Van Holten, Preprint Wuppertal, WUB 84-3 (1984).
2) F. Ernst, Phys. Rev. **167** (1968) 1175.
D. Maison, MPI-PAE/PPh 14/78, München (1978).
3) E. Cremmer and B. Julia, Phys. Lett. **80B** (1978) 48; Nucl. Phys. **B159** (1979) 141.
J. Ellis, M.K. Gaillard, M. Günaydin and B. Zumino, Nucl. Phys. **B224** (1983) 427.
4) F.J. Wegner, Z. Phys. **B35** (1979) 207.
S. Hikami, Phys. Rev. B **24** (1981) 2671.
A. Pruisken and L. Schaffer, Nucl. Phys. **B200** (1982) 20.
5) A.A. Zheltukhin, Theor. Mat. Fiz. **52** (1982) 73.
V.V. Nesterenko, Lett. Math. Phys. **7** (1983) 287.
6) A.M. Kosevich, B.A. Ivanov and A.S. Kovalev, Nonlinear Waves of Magnetization. Dynamical and Topological Solitons (Naukova Dumka, Kiev, 1983) (in Russian).
7) L.A. Takhtajan, Phys. Lett. **64A** (1977) 235.
8) V.E. Zakharov and L.A. Takhtajan, Theor. Mat. Fiz. **38** (1979) 26.
9) M. Lakshmanan, Phys. Lett **61A** (1977) 53.
10) A. Kundu and O.K. Pashaev, J. Phys. C: Solid State Phys. **16** (1983) L585.
11) V.P. Kotlyarov, Doklady AN UkrSSR **A10** (1981) 9.
12) G.R. Quispel and H.W. Capel, Phys. Lett. **88A** (1982) 371.
13) K. Nakamura and T. Sasada, J. Phys. C: Solid State Phys. **15** (1982) L915.
14) O.K. Pashaev, JINR 2-83-230 (Dubna, 1983).
15) V.G. Makhankov, O.K. Pashaev and S.A. Sergeenkov, JINR P2-84-513 (Dubna, 1984).
16) V.P. Kotlyarov, J. Phys. C: Solid State Phys. **17** (1984) L139.
17) V.G. Makhankov and O.K. Pashaev, Phys. Lett. **95A** (1983) 95.
18) V.G. Makhankov and O.K. Pashaev in: Group-Theoretical Methods in Physics, vol. II (Nauka, Moscow, 1983).
19) J. Des Cloizeaux and J.J. Pearson, Phys. Rev. **128** (1962) 2131.
P.W. Anderson, Phys. Rev. **86** (1952) 694.
20) V.G. Makhankov, O.K. Pashaev and S.A. Sergeenkov, JINR Rapid Comm. **10** (1985) 45.
V.G. Makhankov and O.K. Pashaev, JINR P17-85-565 (Dubna, 1985).
21) N.N. Bogolubov, Izv. Akad. Nauk SSSR. Ser. Fiz. **11** (1947) 17.
22) V.E. Zakharov and A.B. Shabat, ZhETF **64** (1973) 1627.
23) V.G. Makhankov, O.K. Pashaev and S.A. Sergeenkov, JINR P2-83-186 (Dubna, 1983).
24) S.T. Belyaev, in: The Many Body Problem, C. De Witt, ed. (Wiley, New York, 1958).
25) H.C. Fogedby, J. Phys. A: Math. Gen **13** (1980) 1467.
26) M. Lakshmanan, T.W. Ruijgrok and C.J. Thompson, Physica **84A** (1976) 577.
27) C.N. Yang and C.P. Yang, Phys. Rev. **150** (1966) 321.
28) E.G. Batiyev and L.S. Braginsky, ZhETF **87** (1984) 1361.
29) L.D. Faddeev and L.A. Takhtajan, in: Differential geometry, Lie Groups and Mechanics. IV, Zap. nauch. semin. LOMI, vol. 109 (Nauka, Leningrad, 1981) 134.