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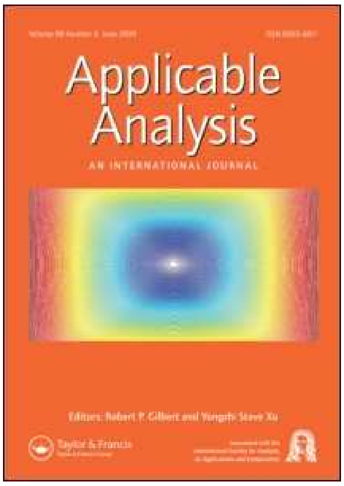
2 authors:

Michal Feckan
Comenius University in Bratislava

Vassilis Rothos
Aristotle University of Thessaloniki



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Michal Fečkan[ab]; Vassilis M. Rothos[c]
[a] Department of Mathematical Analysis and Numerical Mathematics, Comenius University, 842 48 Bratislava, Slovakia [b] Mathematical Institute of Slovak Academy of Sciences, 814 73 Bratislava, Slovakia [c] Faculty of Engineering, School of Mathematics, Physics and Computational Sciences, Aristotle University of Thessaloniki, Thessaloniki 54124, Greece

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Travelling waves of discrete nonlinear Schrödinger equations with nonlocal interactions

Michal Fečkan[ab]* and Vassilis M. Rothos[c]

[a]*Department of Mathematical Analysis and Numerical Mathematics, Comenius University, Mlynská dolina, 842 48 Bratislava, Slovakia;* [b]*Mathematical Institute of Slovak Academy of Sciences, Štefánikova 49, 814 73 Bratislava, Slovakia;* [c]*Faculty of Engineering, School of Mathematics, Physics and Computational Sciences, Aristotle University of Thessaloniki, Thessaloniki 54124, Greece*

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Existence and bifurcation results are derived for quasi periodic travelling waves of discrete nonlinear Schrödinger equations with nonlocal interactions and with polynomial-type potentials. Variational tools are used. Several concrete nonlocal interactions are studied as well.

Keywords: nonlocal interactions; discrete Schrödinger equation; travelling wave; symmetry

AMS Subject Classifications: 34K14; 37K60; 37L60

1. Introduction

One of the most exciting areas in applied mathematics is the study of the dynamics associated with the propagation of information. Coherent structures like solitons, kinks, vortices, etc. play a central role, as carriers of energy, in many nonlinear physical systems [1]. Solitons represent a rare example of a (relatively) recently arisen mathematical object which has found successful high-technology applications [2]. The nature of the system dictates that the relevant and important effects occur along one axial direction. Interplay between nonlinearity and periodicity is the focus of recent studies in different branches of modern applied mathematics and nonlinear physics. Applications range from nonlinear optics, in the dynamics of guided waves in inhomogeneous optical structures and photonic crystal lattices, to atomic physics, in the dynamics of Bose–Einstein condensate (BEC) droplets in periodic potentials, and from condensed matter, in Josephson-junction ladders, to biophysics, in various models of the DNA double strand. Analysis and modelling of these physical situations are based on nonlinear evolution equations derived from underlying physics equations, such as nonlinear Maxwell equations with periodic coefficients [3]. In particular, the systems of 2nd-order nonlinear Schrödinger (NLS) equations, both

*Corresponding author. Email: Michal.Feckan@fmph.uniba.sk

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continuous and discrete, were applied in nonlinear physics to study a number of experimental and theoretical problems. Spatial nonlocality of the nonlinear response is also naturally present in the description of BECs where it represents the finite range of the bosonic interaction. Demands on the mathematics for techniques to analyse these models may best be served by developing methods tailored to determining the local behaviour of solutions near these structures. The discreteness of space, i.e. the existence of an underlying spatial lattice is crucial to the structural stability of these spatially localized nonlinear excitations.

During the early years, the studies of intrinsic localized modes were mostly of a mathematical nature, but the ideas of localized modes soon spread to theoretical models of many different physical systems, and the discrete breather concept has been recently applied to experiments in several different physics subdisciplines. Most nonlinear lattice systems are not integrable even if the partial differential equation (PDE) model is in the continuum limit. While for many years spatially continuous nonlinear PDEs and their localized solutions have received a great deal of attention, there has been increasing interest in spatially discrete nonlinear systems. Namely, the dynamical properties of nonlinear systems based on the interplay between discreteness, nonlinearity and dispersion (or diffraction) can find wide applications in various physical, biological and technological problems. Examples are coupled optical fibres (self-trapping of light) [4–7], arrays of coupled Josephson junctions [8], nonlinear charge and excitation transport in biological macromolecules, charge transport in organic semiconductors [9].

Prototype models for such nonlinear lattices take the form of various nonlinear lattices [10], a particularly important class of solutions, of which, are the so-called discrete breathers which are homoclinic in space and oscillatory in time. Other questions involve the existence and propagation of topological defects or kinks which mathematically are heteroclinic connections between a ground and an excited steady state. Prototype models here are discrete version of sine-Gordon equations, also known as Frenkel–Kontorova (FK) models, e.g. [11]. There are many outstanding issues for such systems relating to the global existence and dynamics of localized modes for general nonlinearities, away from either continuum or anti-continuum limits.

In the main part of the previous studies of the discrete NLS models, the dispersive interaction was assumed to be short-ranged and a nearest-neighbour approximation was used. However, there exist physical situations that definitely cannot be described in the framework of this approximation. The DNA molecule contains charged groups, with long-range Coulomb interaction $1/r$ between them. The excitation transfer in molecular crystals [12] and the vibron energy transport in biopolymers [13] are due to transition dipole–dipole interaction with $1/r^3$ dependence on the distance, r. The nonlocal (long-range) dispersive interaction in these systems provides the existence of additional length-scale: the radius of the dispersive interaction. We will show that it leads to the bifurcating properties of the system due to both the competition between nonlinearity and dispersion, and the interplay of long-range interactions and lattice discreteness.

In some approximation, the equation of motion is the nonlocal discrete NLS

$$\imath \dot{u}_n = \sum_{m \neq n} J_{n-m}(u_n - u_m) + |u_n|^2 u_n, \quad n \in \mathbb{Z}, \tag{1}$$

where the long-range dispersive coupling is taken to be either exponentially $J_n = Je^{-\beta|n|}$ with $\beta > 0$, or algebraically $J_n = J|n|^{-s}$ with $s > 0$, decreasing with the distance n between lattice sites. In both cases the constant J is normalized such that $\sum_{n=1}^{\infty} J_n = 1$, for all β or s. The parameters β and s are introduced to cover different physical situations from the nearest-neighbour approximation ($\beta \to \infty$, $s \to \infty$) to the quadrupole–quadrupole ($s = 5$) and dipole–dipole ($s = 3$) interactions. The Hamiltonian H and the number of excitations N

$$H = \frac{1}{2}\sum_{n,m\in\mathbb{Z}} J_{n-m}|u_n - u_m|^2 - \frac{1}{2}\sum_{n\in\mathbb{Z}} |u_n|^4 \quad \text{and} \quad N = \sum_{n\in\mathbb{Z}} |u_n|^2 \tag{2}$$

are conserved quantities corresponding to the set of (1).

It should also be noted that the derivation of a discrete equation from the Gross–Pitaevskii equation produces at the intermediate step a fully nonlocal discrete NLS equation for the coefficients of the wave function expansion over the complete set of the Wannier functions. Further reduction to the case of the only band with the strong localization of the Wannier functions (the tight-binding approximation) leads to the standard local discrete nonlinear Schrödinger (DNLS) equation. Recently Abdullaev et al. [14] extended this approach to the case of periodic nonlinearities and derived a number of nonintegrable lattices with different nearest-neighbour nonlinearities.

In this article, we study the DNLS equations on the lattice $\mathbb{Z}$ (DNLS) with nonlocal interactions of forms

$$\imath\dot{u}_n = \sum_{j\in\mathbb{N}} a_j \Delta_j u_n + f(|u_n|^2)u_n, \quad n \in \mathbb{Z}, \tag{3}$$

where $u_n \in \mathbb{C}$, $\Delta_j u_n := u_{n+j} + u_{n-j} - 2u_n$ are 1-dimensional discrete Laplacians and it holds

(H1) $f \in C(\mathbb{R}_+, \mathbb{R})$ for $\mathbb{R}_+ := [0, \infty)$, $f(0) = 0$ and $a_j \in \mathbb{R}$ with $\sum_{j\in\mathbb{N}} |a_j| < \infty$. Moreover, there are constants $s > 0$, $\mu > 1$, $c_1 > 0$, $c_2 > 0$ and $\bar{r} > 0$ such that

$$|f(w)| \le c_1(w^s + 1), \quad c_2(w^{s+1} - 1) \le F(w), \quad \mu F(w) - \bar{r} < f(w)w$$

for any $w \ge 0$, where $F(w) = \int_0^w f(z)\mathrm{d}z$. Furthermore, $\limsup_{w\to 0_+} f(w)/w^{\widetilde{s}} < \infty$ for a constant $\widetilde{s} > 0$.

Of course, we suppose that not all a_j are zero. Note any polynomial $f(w) = p_1 w + \cdots + p_s w^s$, $s \in \mathbb{N}$ with $p_s > 0$ satisfies (H1). Furthermore, (3) can be rewritten into a standard form

$$\imath\dot{u}_n = \sum_{m\neq n} a_{|m-n|}(u_m - u_n) + f(|u_n|^2)u_n, \quad n \in \mathbb{Z}. \tag{4}$$

It is well known that (4) conserves two dynamical invariants

$$\sum_{n\in\mathbb{Z}} |u_n|^2 - \text{the norm},$$

$$\sum_{n\in\mathbb{Z}} \left[-\frac{1}{2}\sum_{m\neq n} a_{|m-n|}|u_m - u_n|^2 + F(|u_n|^2) \right] - \text{the energy}.$$

Differential equations with nonlocal interactions on lattices have been studied in [15–23], while DNLS in [20,24–27]. Nowadays it is clear that a large number of important models of various fields of physics are based on DNLS type equations with several forms of polynomial nonlinearities starting with the simplest self-focusing cubic (Kerr) nonlinearity, then following with the cubic onsite nonlinearity relevant for BECs, and then with more general discrete cubic nonlinearity in Salerno model up to cubic-quintic ones (see [25] for more references).

We are interested in the existence of travelling wave solutions $u_n(t)=U(n-\nu t)$ of (3) with a quasi periodic function $U(z)$, $z=n-\nu t$ and some $\nu\neq 0$.

First, we introduce a function

$$\Phi(x):=\frac{4}{x}\sum_{j\in\mathbb{N}}a_j\sin^2\Big[\frac{x}{2}j\Big].$$

Remark 1 Clearly $\Phi\in C(\mathbb{R}\setminus\{0\},\mathbb{R})$, Φ is odd, $\Phi(2\pi k)=0$ for any $k\in\mathbb{Z}\setminus\{0\}$ and $\Phi(x)\to 0$ as $|x|\to\infty$. If $\sum_{j\in\mathbb{N}}j\,|a_j|<\infty$ then $\Phi\in C(\mathbb{R},\mathbb{R})$ and if $\sum_{j\in\mathbb{N}}j^2|a_j|<\infty$ then $\Phi\in C^1(\mathbb{R},\mathbb{R})$. Consequently the range $\mathcal{R}\Phi:=\Phi(\mathbb{R}\setminus\{0\})$ is either an interval $[-\bar{R},\bar{R}]$ or $(-\bar{R},\bar{R})$ here with possibility $\bar{R}=\infty$ (see Section 2.4 for concrete examples).

Now we can state the following existence result.

THEOREM 1.1 *Let* (H1) *hold and* $T>0$. *Then for almost each* $\nu\in\mathbb{R}\setminus\{0\}$ *and any rational* $r\in\mathbb{Q}\cap(0,1)$, *there is a nonzero periodic travelling wave solution* $u_n(t)=U(n-\nu t)$ *of* (3) *with* $U\in C^1(\mathbb{R},\mathbb{C})$ *and such that*

$$U(z+T)=\mathrm{e}^{2\pi r\imath}U(z),\quad \forall z\in\mathbb{R}. \tag{5}$$

Moreover, for any $\nu\in\mathbb{R}\setminus\{0\}$ *there is at most a finite number of* $\bar{r}_1,\bar{r}_2,\dots,\bar{r}_m\in(0,1)$ *such that equation*

$$-\nu=\Phi\left(\frac{2\pi}{T}(\bar{r}_j+k)\right)$$

has a solution $k\in\mathbb{Z}$. *Then for any* $r\in(0,1)\setminus\{\bar{r}_1,\bar{r}_2,\dots,\bar{r}_m\}$ *there is a nonzero quasi periodic travelling wave solution* $u_n(t)=U(n-\nu t)$ *with the above properties. In particular, for any* $|\nu|>\bar{R}$ *and* $r\in(0,1)$, *there is such a nonzero quasi periodic travelling wave solution.*

When a nonresonance condition of Theorem 1.1 fails, then we have the following bifurcation results.

THEOREM 1.2 *Suppose* $f\in C^1(\mathbb{R}_+,\mathbb{R})$ *with* $f(0)=0$. *If there are* $\bar{r}_1\in(0,1)$, $\nu\in\mathcal{R}\Phi\setminus\{0\}$ and $T>0$ *such that all solutions* $k_1,k_2,\dots,k_{m_1}\in\mathbb{Z}$ *of equation*

$$-\nu=\Phi\left(\frac{2\pi}{T}(\bar{r}_1+k)\right)$$

are either nonnegative or negative and $m_1>0$. *Then for any* $\varepsilon>0$ *small there are* m_1 *branches of nonzero quasi periodic travelling wave solutions* $u_{n,j,\varepsilon}(t)=U_{j,\varepsilon}(n-\nu_\varepsilon t)$ *of* (3) *with* $U_{j,\varepsilon}\in C^1(\mathbb{R},\mathbb{C})$, $j=1,2,\dots,m_1$, *and nonzero velocity* ν_ε *satisfying* $U_{j,\varepsilon}(z+T)=\mathrm{e}^{2\pi\bar{r}_1\imath}U_{j,\varepsilon}(z)$, $\forall z\in\mathbb{R}$ *along with* $\nu_\varepsilon\to\nu$ *and* $U_{j,\varepsilon}\rightrightarrows 0$ *uniformly on* $\mathbb{R}$ *as* $\varepsilon\to 0$.

Remark 2 If $a_j \geq 0$ for all $j \in \mathbb{N}$, then the assumptions of Theorem 1.2 are satisfied for any $\nu \in \mathcal{R}\Phi \setminus \{0\}$ such that $\frac{T}{2\pi}\Phi^{-1}(-\nu) \setminus \mathbb{Z} \neq \emptyset$, and so there are bifurcations of quasi periodic travelling waves in the generic resonant cases. On the other hand, if $\nu \in \mathcal{R}\Phi \setminus \{0\}$ with $\frac{T}{2\pi}\Phi^{-1}(-\nu) \subset \mathbb{Z}$ then Theorem 1.1 is applicable for any $r \in (0, 1)$.

Theorem 1.2 is a Lyapunov centre theorem for travelling wave solutions. Similar results are derived in [28] for Fermi–Pasta–Ulam lattices.

We also discuss in Section 4 the extension of these results of (3) on the lattices $\mathbb{Z}^2$ and $\mathbb{Z}^3$ [24–27]. Section 5 is devoted to travelling wave solutions of more general forms than above [29]. Finally, related results are also presented in [30–37].

2. Existence of travelling wave solutions

In this section, we study the existence of travelling wave solutions of the form $u_n(t) = U(n - \nu t)$, i.e. we are interested in the equation

$$-\nu\imath U'(z) = \sum_{j\in\mathbb{N}} a_j \partial_j U(z) + f(|U(z)|^2)U(z), \tag{6}$$

where $z = n - \nu t$, $\nu \neq 0$ and $\partial_j U(z) := U(z + j) + U(z - j) - 2U(z)$. We are interested in the existence of quasi periodic solutions $U(z)$ of (6) stated in Theorem 1.1.

2.1. *Preliminaries*

In this subsection we recall some results from critical point theory of [38]. Let H be a Hilbert space and let $J \in C^1(H, \mathbb{R})$. Suppose $H = H_1 \oplus H_2$ for closed linear subspaces, and let $e_1, e_2, \ldots$ be the orthonormal basis of H_1. Let us put $H_n^1 := \operatorname{span}\{e_1, e_2, \ldots, e_n\}$ and $H_n := H_n^1 \oplus H_2$. Let P_n be the orthogonal projection of H onto H_n. Set $J_n := J/H_n$ – the restriction of functional J on subspace H_n – and so $\nabla J_n(x) = P_n \nabla J(x)$ if $x \in H_n$.

Definition 2.1 If there are two positive constants α and β such that

$$\begin{aligned}
J(x) &\geq 0 \quad \forall x \in \{x \in H_1 \mid \|x\| \leq \beta\},\\
J(x) &\geq \alpha \quad \forall x \in \{x \in H_1 \mid \|x\| = \beta\},\\
J(x) &\leq 0 \quad \forall x \in \{x \in H_2 \mid \|x\| \leq \beta\},\\
J(x) &\leq -\alpha \quad \forall x \in \{x \in H_2 \mid \|x\| = \beta\},
\end{aligned}$$

then J is said to satisfy the local linking condition at 0.

Definition 2.2 We shall say that J satisfies the Palais–Smale (PS)*-condition if any sequence $\{x_n\}_{n\in\mathbb{N}}$ in H such that $x_n \in H_n$, $J(x_n) \leq c < \infty$ and $P_n \nabla J(x_n) = \nabla J_n(x_n) \to 0$ as $n \to \infty$ possesses a convergent subsequence.

Now we can state the following theorem of [38] which we apply.

THEOREM 2.3 *Suppose*

(I_1) $J \in C^1(H, \mathbb{R})$ *satisfies* $(PS)^*$*-condition.*
(I_2) J *satisfies the local linking condition at* 0.

(I_3) $\forall n$, $J_n(x) \to -\infty$ *as* $\|x\| \to \infty$ *and* $x \in H_n$.
(I_4) $\nabla J = A + C$ *for a bounded linear self-adjoint operator* A *such that* $AH_n \subset H_n$, $\forall n \in \mathbb{N}$ *and* C *is a compact mapping.*

Then J *possesses a critical point* $\bar{x}$ *with* $|J(\bar{x})| \geq \alpha$.

Remark 1 If 0 is an indefinite nondegenerate critical point of J, then J satisfies the local linking condition at 0.

2.2. *Proof of Theorem 1.1*

In this section, we use Theorem 2.3 to prove Theorem 1.1. Without loss of generality, we set $T = 2\pi$. We suppose $\nu > 0$, the case $\nu < 0$ can be handled similarly. First, we identify $\mathbb{C}$ with $\mathbb{R}^2$ in this section. Let $r \in (0,1)$ be fixed. Next, we consider real Banach spaces

$$L_r^{\widetilde{s}} := \left\{ U \in L_{\mathrm{loc}}^{\widetilde{s}}(\mathbb{R}, \mathbb{C}) \mid U(z + 2\pi) = \mathrm{e}^{2\pi r \imath} U(z),\ \forall z \in \mathbb{R} \right\}$$

for $\widetilde{s} \geq 1$. Clearly $U \in L_r^{\widetilde{s}}$ if and only if $U(z) = \mathrm{e}^{rz\imath} V(z)$ for some $V \in L^{\widetilde{s}} := L^{\widetilde{s}}(S^{2\pi}, \mathbb{C})$. Consequently $U_1(z + c_1)\overline{U_2(z + c_2)}$ is 2π-periodic for any $c_1, c_2 \in \mathbb{R}$ and $U_1, U_2 \in L_r^{\widetilde{s}}$, hence $|U(z)|$ is 2π-periodic. So we consider the norm on $L_r^{\widetilde{s}}$ like on $L^{\widetilde{s}}$. In particular, we have

$$V \in L_r^2 \Leftrightarrow V(z) = \sum_{k \in \mathbb{Z}} V_k\, \mathrm{e}^{(r+k)z\imath}, \quad V_k \in \mathbb{C}, \quad \sum_{k \in \mathbb{Z}} |V_k|^2 < \infty.$$

Let

$$X_r := W_r^{1/2,2}(S^{2\pi}, \mathbb{C}) = \left\{ V \in L_r^2 \mid V(z) = \sum_{k \in \mathbb{Z}} V_k\, \mathrm{e}^{(r+k)z\imath},\ \sum_{k \in \mathbb{Z}} |V_k|^2 |r + k| < \infty \right\},$$

$$Y_r := W_r^{1,2}(S^{2\pi}, \mathbb{C}) = \left\{ V \in L_r^2 \mid V(z) = \sum_{k \in \mathbb{Z}} V_k\, \mathrm{e}^{(r+k)z\imath},\ \sum_{k \in \mathbb{Z}} |V_k|^2 (r + k)^2 < \infty \right\}.$$

Note $r + k \neq 0$ for any $k \in \mathbb{Z}$. Clearly $Y_r \subset X_r \subset L_r^2$. We consider L_r^2, X_r and Y_r as real Hilbert spaces with inner products

$$\langle V, W \rangle_{L_r^2} := 2\pi \Re \sum_{k \in \mathbb{Z}} V_k \overline{W_k} = \Re \int_0^{2\pi} V(z)\overline{W(z)}\mathrm{d}z,$$
$$\langle V, W \rangle_{X_r} := 2\pi \Re \sum_{k \in \mathbb{Z}} V_k \overline{W_k} |r + k|$$
$$\langle V, W \rangle_{Y_r} := 2\pi \Re \sum_{k \in \mathbb{Z}} V_k \overline{W_k} (r + k)^2$$

for $V(z) = \sum_{k \in \mathbb{Z}} V_k\, \mathrm{e}^{(r+k)z\imath}$ and $W(z) = \sum_{k \in \mathbb{Z}} W_k\, \mathrm{e}^{(r+k)z\imath}$.

Clearly $\|U\|_{L^2} = \|U\|_{L_r^2} \leq r_1 \|U\|_{X_r}$, $\forall U \in X_r$ and $\|U\|_{Xr} \leq r_1 \|U\|_{Yr} = r_1 \|U'\|_{L^2}$, $\forall U \in Y_r$ for $r_1 := \min\left\{\sqrt{r}, \sqrt{1 - r}\right\}$. The following result is well known [38,39].

Lemma 2.4 *For each* $\widetilde{s} \geq 1$, X_r *is compactly embedded into* $L_r^{\widetilde{s}}$.

On X_r, we consider a continuous symmetric bilinear form

$$B_r(U, V) := 4\pi \Re \sum_{k \in \mathbb{Z}} V_k \overline{W_k} (r + k).$$

Note, if $U \in X_r$ and $V \in Y_r$, then

$$2\Re \int_0^{2\pi} \imath U(z)\overline{V(z)}'\mathrm{d}z = B_r(U, V).$$

Now we consider a real functional

$$\begin{aligned} I_r(U) &:= \frac{\nu}{2} B_r(U, U) + \int_0^{2\pi} \left\{ \sum_{j \in \mathbb{Z}} \frac{a_{|j|}}{2} |U(z+j) - U(z)|^2 - F(|U(z)|^2) \right\} \mathrm{d}z \\ &= \frac{\nu}{2} B_r(U, U) + \int_0^{2\pi} \left\{ \sum_{j \in \mathbb{N}} a_j |U(z+j) - U(z)|^2 - F(|U(z)|^2) \right\} \mathrm{d}z \end{aligned}$$

on X_r. Then $I_r \in C^1(X_r, \mathbb{R})$ and for $U \in X_r$, $V \in Y_r$, we derive

$$DI_r(U)V = 2\Re \left\{ \int_0^{2\pi} \left(\nu\imath U(z)\overline{V(z)}' - \left(\sum_{j \in \mathbb{N}} a_j \partial_j U(z) + f(|U(z)|^2)U(z) \right) \overline{V(z)} \right) \mathrm{d}z \right\}.$$

If $U \in X_r$ is a critical point of I_r then

$$\Re \left\{ \int_0^{2\pi} \left(\nu\imath U(z)\overline{V(z)}' - \left(\sum_{j \in \mathbb{N}} a_j \partial_j U(z) + f(|U(z)|^2)U(z) \right) \overline{V(z)} \right) \mathrm{d}z \right\} = 0 \qquad (7)$$

for any $V \in Y_r$. Replacing V with $\imath V$ in (7), we obtain

$$\int_0^{2\pi} \left(\nu\imath U(z)\overline{V(z)}' - \left(\sum_{j \in \mathbb{N}} a_j \partial_j U(z) + f(|U(z)|^2)U(z) \right) \overline{V(z)} \right) \mathrm{d}z = 0$$

for any $V \in Y_r$. This means that U is a weak solution of (6). Then a standard regularity method shows [39] that U is a C^1-smooth solution of (6).

Now we split $X_r = X_+ \oplus X_-$ for

$$X_- := \left\{ V(z) = \sum_{k=-\infty}^{-1} V_k \, \mathrm{e}^{(r+k)z\imath} \right\}, \quad X_+ := \left\{ V(z) = \sum_{k=0}^{\infty} V_k \, \mathrm{e}^{(r+k)z\imath} \right\}.$$

Clearly if $U = U_+ + U_-$ then $B_r(U, U) = 2\left(\|U_+\|_{X_r}^2 - \|U_-\|_{X_r}^2 \right)$.

Next, let us define $\widetilde{K}_r : L_r^2 \to X_r$ as

$$\langle \widetilde{K}_r H, V \rangle_{X_r} := 2\Re \int_0^{2\pi} H(z)\overline{V(z)}\mathrm{d}z, \quad \forall V \in X_r. \qquad (8)$$

Then

$$\widetilde{K}_r H = \sum_{k \in \mathbb{Z}} \frac{2H_k}{|r+k|} \, \mathrm{e}^{(r+k)\imath z}$$

and so $\widetilde{K}_r$ is compact. To study $\nabla I_r(u)$, we introduce the mapping $\Psi_r : X_r \to X_r$ defined by

$$\langle \Psi_r(U), V \rangle_{X_r} := 2\Re \int_0^{2\pi} f(|U(z)|^2)U(z)\overline{V(z)}\mathrm{d}z$$

for any $V \in X_r$. By Lemma 2.4, the Nemytskij operator $U \to f(|U(z)|^2)U(z)$ from X_r to L_r^2 is continuous. Using (8), we get

$$\Psi_r(U) = \widetilde{K}_r f(|U|^2)U.$$

Hence $\Psi_r : X_r \to X_r$ is compact and continuous.

LEMMA 2.5 *Under* (H1) *it holds* $D\Psi_r(0) = 0$.

proof There is a constant c_3 such that

$$|f(w)| \le c_3(w + w^s)$$

for any $w \ge 0$. Then by Lemma 2.4, we derive

$$\begin{aligned} |f(|U|^2)U|_{L_r^2}^2 &= \int_0^{2\pi} f(|U(z)|^2)^2 |U(z)|^2 \mathrm{d}z \\ &\le 2c_3^2 \int_0^{2\pi} \left(|U(z)|^6 + |U(z)|^{2(2s+1)}\right)\mathrm{d}z \le c_4^2\left(\|U\|_{X_r}^3 + \|U\|_{X_r}^{2s+1}\right)^2 \end{aligned}$$

for a constant $c_4 > 0$. Hence

$$\left|\langle \Psi_r(U), V\rangle_{X_r}\right| \le 2\|f(|U|^2)U\|_{L_r^2}\|V\|_{L_r^2} \le c_5\left(\|U\|_{X_r}^3 + \|U\|_{X_r}^{2s+1}\right)\|V\|_{X_r}$$

for a constant $c_5 > 0$. This implies

$$\|\Psi_r(U)\|_{X_r} \le c_5\left(\|U\|_{X_r}^3 + \|U\|_{X_r}^{2s+1}\right), \quad \forall U \in X_r.$$

Since $\Psi_r(0) = 0$ and $s > 0$, we get $D\Psi_r(0) = 0$. The proof is finished. ∎

Finally, define $\mathcal{L}_r : L_r^2 \to L_r^2$ as

$$\mathcal{L}_r U := \sum_{j \in \mathbb{N}} a_j \partial_j U(z).$$

Then

$$\nabla I_r(U) = \left(2\nu \boldsymbol{I}_+ - 2\nu \boldsymbol{I}_- - \widetilde{K}_r \mathcal{L}_r - \Psi_r\right)(U) \tag{9}$$

for the identities $\boldsymbol{I}_\pm : X_\pm \to X_\pm$. Clearly

$$A_r := 2\nu \boldsymbol{I}_+ - 2\nu \boldsymbol{I}_- - \widetilde{K}_r \mathcal{L}_r$$

is a self-adjoint bounded operator $A_r : X_r \to X_r$ satisfying

$$A_r U = 2\sum_{k \in \mathbb{Z}} \left(\nu\, \mathrm{sgn}\,(r+k) + \frac{4}{|r+k|}\sum_{j \in \mathbb{N}} a_j \sin^2\left[\frac{r+k}{2} j\right]\right) U_k\, \mathrm{e}^{(r+k)\imath z}.$$

Consequently, the spectrum $\sigma(A_r)$ of A_r is given by

$$\sigma(A_r) = \left\{2\,\mathrm{sgn}\,(r+k)(\nu + \Phi(r+k)) \mid k \in \mathbb{Z}\right\}.$$

By Lemma 2.5, we get that under the assumption

$$-\nu \neq \Phi(r+k) \quad \forall k \in \mathbb{Z}, \tag{10}$$

0 is an indefinite nondegenerate critical point of I_r: $\nabla I_r(0) = 0$ and Hess $I_r(0) = A_r$ with $0 \notin \sigma(A_r)$ and $X_r = X_{1,r} \oplus X_{2,r}$ with $\sigma(A_r/X_{1,r}) \subset (0, \infty)$ and $\sigma(A_r/X_{2,r}) \subset (-\infty, 0)$ where $X_{1,r}$, $X_{2,r}$ are suitable closed linear subspaces of X_r. Note $X_{1,r}$ and $X_{2,r}$ are infinite dimensional, since $\Phi(r+k) \to 0$ as $|k| \to \infty$. Consequently by Remark 1, under (10), I_r satisfies the local linking condition at 0 in the sense of Definition 2.1, i.e. condition (I_2) of Theorem 2.3 is verified.

We consider an equivalent scalar product $\langle \cdot, \cdot \rangle_r$ on X_r such that

$$\langle A_r U, U \rangle_r = \|U_1\|_r^2 - \|U_2\|_r^2, \quad U_1 \in X_{1,r},\ U_2 \in X_{2,r}.$$

Note there is a linear isomorphism $K_r : X_r \to X_r$ such that

$$\langle U, V \rangle_{X_r} = \langle K_r U, V \rangle_r, \quad \forall U, \forall V \in X_r.$$

Clearly K_r is self-adjoint and positive definite. Then

$$\begin{aligned}
I_r(U) &= \frac{\nu}{2}\|U_1\|_r^2 - \frac{\nu}{2}\|U_2\|_r^2 - \int_0^{2\pi} F(|U(z)|^2)\mathrm{d}z, \\
\nabla I_r(U) &= \nu \boldsymbol{I}_1 - \nu \boldsymbol{I}_2 - K_r \Psi_r, \\
\langle \nabla I_r(U), V \rangle_r &= DI_r(U)V = \nu\|V_1\|_r^2 - \nu\|V_2\|_r^2 \\
&\quad - 2\Re \int_0^{2\pi} f(|U(z)|^2)U(z)\overline{V(z)}\mathrm{d}z.
\end{aligned}$$

Let $X_{1,r} = \mathrm{span}\{e_1, e_2, \ldots\}$ and e_i are eigenvectors of A_r. Then we take $X_n = \mathrm{span}\{e_1, e_2, \ldots, e_n\} \oplus X_{2,r}$ for $n \geq 3$. So clearly $A_r X_n \subset X_n$, i.e. condition (I_4) of Theorem 2.3 is verified. Let $P_n : X_r \to X_n$ be the orthogonal projection with respect to $\langle \cdot, \cdot \rangle_r$.

We suppose there is a sequence $\{U_m\}_{m \in \mathbb{N}} \subset X_r$, $U_m \in X_m$ and a constant c such that

$$I_r(U_m) \leq c \quad \text{and} \quad P_m \nabla I_r(U_m) \to 0.$$

Then for m large we get,

$$\begin{aligned}
c + \|U_m\|_r &\geq I_r(U_m) - \tfrac{1}{2}\langle P_m \nabla I_r(U_m), U_m \rangle_r \\
&= \int_0^{2\pi} \left[f(|U_m(z)|^2)|U_m(z)|^2 - F(|U_m(z)|^2)\right]\mathrm{d}z \\
&\geq \int_0^{2\pi} (\mu - 1)F(|U_m(z)|^2)\mathrm{d}z - 2\pi\bar{r} \\
&\geq (\mu - 1)c_2 \int_0^{2\pi} \left(|U_m(z)|^{2(s+1)} - 1\right)\mathrm{d}z - 2\pi\bar{r} \\
&\geq (\mu - 1)c_2\left(\|U_m\|_{L^{2(s+1)}}^{2(s+1)} - c_6\right)
\end{aligned} \tag{11}$$

for a constant $c_6 > 0$.

By following the same arguments, we derive

$$\begin{aligned}\nu\|U_{1,m}\|_r^2 &\leq \|P_m\nabla I_m(U_m)\|\cdot\|U_{1,m}\|_r + 2\int_0^{2\pi} f(|U_m(z)|^2)|U_m(z)||U_{1,m}(z)|\mathrm{d}z\\ &\leq \|U_{1,m}\|_r + 2c_7\int_0^{2\pi}\big(|U_m(z)|^{2s+1}+1\big)|U_{1,m}(z)|\mathrm{d}z\\ &\leq \|U_{1,m}\|_r + 2c_7\big\||U_m|^{2s+1}+1\big\|_{L^{\frac{2(s+1)}{2s+1}}}\|U_{1,m}\|_{L^{2(s+1)}}\\ &\leq \|U_{1,m}\|_r + 2c_7\big(\|U_m\|_{L^{2(s+1)}}^{2s+1}+1\big)\|U_{1,m}\|_r\end{aligned}$$

and hence

$$\|U_{1,m}\|_r \leq c_8\big(\|U_m\|_{L^{2(s+1)}}^{2s+1}+1\big).$$

Similarly we obtain

$$\|U_{2,m}\|_r \leq c_8\big(\|U_m\|_{L^{2(s+1)}}^{2s+1}+1\big)$$

and consequently by (11), we obtain

$$\|U_m\|_r \leq 2c_8\big(\|U_m\|_{L^{2(s+1)}}^{2s+1}+1\big) \leq c_9\left(\|U_m\|_r^{\frac{2s+1}{2(s+1)}}+1\right)$$

for positive constants c_7, c_8 and c_9. Thus $\{U_m\}_{m\in\mathbb{N}}\subset X_r$ is bounded. Since

$$P_m\nabla I_r(U_m) = \nu U_{1,m} - \nu U_{2,m} - K_r\Psi_r(U_m) \to 0$$

and $K_r\Psi_r$ is compact, there is a convergent subsequence of $\{U_m\}_{m\in\mathbb{N}}$ in X_r. Summarizing, (PS)*-condition is verified for I_r, i.e. condition (I_1) of Theorem 2.3 is verified.

Next, let $U\in X_n$. Then using $U_1\in\operatorname{span}\{e_1,e_2,\ldots,e_n\}$, we derive

$$\begin{aligned}I_r(U) &= \frac{\nu}{2}\big(\|U_1\|_r^2-\|U_2\|_r^2\big) - \int_0^{2\pi} F(|U(z)|^2)\mathrm{d}z\\ &\leq \frac{\nu}{2}\big(\|U_1\|_r^2-\|U_2\|_r^2\big) - c_2\int_0^{2\pi}\big(|U(z)|^{2(s+1)}-1\big)\mathrm{d}z\\ &\leq \frac{\nu}{2}\big(\|U_1\|_r^2-\|U_2\|_r^2\big) - c_2\|U\|_{L^{2(s+1)}}^{2(s+1)} + c_{10}\\ &\leq \frac{\nu}{2}\big(\|U_1\|_r^2-\|U_2\|_r^2\big) - c_{11}\left(\|U_1\|_{L^2}^{2(s+1)}+\|U_2\|_{L^2}^{2(s+1)}\right) + c_{10}\\ &\leq \frac{\nu}{2}\|U_1\|_r^2\big(1-c_{12}\|U_1\|_r^{2s}\big) - \frac{\nu}{2}\|U_2\|_r^2 + c_{10}\end{aligned}$$

for positive constants c_{10}, c_{11} and c_{12}. Now it is clear that $I_r(U)\to-\infty$ as $\|U\|_r\to\infty$, i.e. condition (I_3) of Theorem 2.3 is verified.

Summarizing, under assumptions (H1) and (10), all conditions (I_1)–(I_4) of Theorem 2.3 are verified for I_r. Hence there is a nonzero critical point $U_r\in X_r$ of I_r, which we already know to be a C^1-smooth solution of (6) satisfying (5). Note (10) certainly holds for any $|\nu|>\bar{R}$ and $r\in(0,1)$. Hence the proof of the second part of Theorem 1.1 is finished. To prove the first part, it is enough to observe that the set

$$\big\{\Phi(r+k)\mid r\in\mathbb{Q}\cap(0,1),\quad k\in\mathbb{Z}\big\}$$

is countable, and thus for almost each $\nu \in \mathbb{R} \setminus \{0\}$ and any $r \in \mathbb{Q} \cap (0, 1)$, condition (10) holds.

2.3. *Remarks*

Remark 2 When r is rational in Theorem 1.1 then we get periodic $U(z)$ with arbitrarily large minimal periods. If r is irrational then clearly $U(z) = \mathrm{e}^{\frac{2\pi}{T} r z \imath} V(z)$ for a T-periodic $V(z) = U(z)\,\mathrm{e}^{-\frac{2\pi}{T} r z \imath}$. So $U(z)$ is quasi periodic and its orbit in $\mathbb{C} \simeq \mathbb{R}^2$ is dense either in a compact annulus or in a compact disc. But $|U(z)|$ is T-periodic in both the cases.

Remark 3 Changing $t \leftrightarrow -t$, we can also handle DNLS

$$-\imath \dot{u}_n = \sum_{j \in \mathbb{N}} a_j \Delta_j u_n + f(|u_n|^2) u_n, \quad n \in \mathbb{Z} \tag{12}$$

under (H1) and (10) becomes

$$\nu \neq \Phi(r + k) \quad \forall k \in \mathbb{Z} \quad \text{and} \quad \nu \in (0, \bar{R}). \tag{13}$$

Remark 4 Assume that $U \in Y_r$ is a weak solution of (6), then

$$\begin{aligned}
|U(z)| &\leq \sum_{k \in \mathbb{Z}} |U_k| \leq \sqrt{\sum_{k \in \mathbb{Z}} |U_k|^2 (r + k)^2} \sqrt{\sum_{k \in \mathbb{Z}} (r + k)^{-2}} \\
&= \sqrt{\frac{\pi}{2}} \operatorname{cosec} \pi r \, \|U\|_{Y_r}.
\end{aligned}$$

Let $\widetilde{R} := \max_{x \in \mathbb{R}_+} x\Phi(x)$. Then

$$\begin{aligned}
|\nu| \|U'\|_{L^2} = |\nu| \|U\|_{Y_r} &\leq \|\mathcal{L}_r U\|_{L^2} + \|f(|U|^2 U)\|_{L^2} \\
&\leq \widetilde{R} \|U\|_{L^2} + c_1 \left\| |U|^{2s+1} + |U| \right\|_{L^2} \\
&\leq \left(\widetilde{R} r_1^2 + c_1 r_1^2 + c_1 \frac{\pi^{s+1}}{2^s} \operatorname{cosec}^{2s+1} \pi r \, \|U\|_{Y_r}^{2s} \right) \|U\|_{Y_r}.
\end{aligned}$$

So if $U \neq 0$ then we obtain

$$|\nu| \leq \widetilde{R} r_1^2 + c_1 r_1^2 + c_1 \frac{\pi^{s+1}}{2^s} \operatorname{cosec}^{2s+1} \pi r \, \|U\|_{Y_r}^{2s},$$

i.e.

$$\|U\|_{Y_r} \geq \sqrt{2} \sqrt[2s]{\frac{|\nu| - \widetilde{R} r_1^2 - c_1 r_1^2}{c_1 \pi^{s+1} \operatorname{cosec}^{2s+1} \pi r}}$$

for

$$|\nu| \geq \widetilde{R} r_1^2 + c_1 r_1^2.$$

Hence $\|U\|_{Y_r} \to \infty$ as $|\nu| \to \infty$ for a possible nonzero solution $U \in Y_r$ of (6).

2.4. *Examples*

We first note

$$\Phi(x) = \frac{2}{x}\sum_{j\in\mathbb{N}} a_j(1-\cos xj) = \frac{2}{x}\left[\sum_{j\in\mathbb{N}} a_j - \Re\sum_{j\in\mathbb{N}} a_j\, \mathrm{e}^{xj\imath}\right]. \tag{14}$$

Now we turn to the following concrete examples.

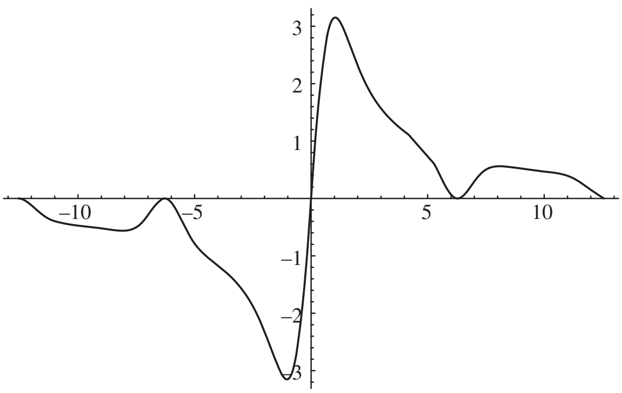
Example 2.6 First we suppose that a_j is decaying rapidly to 0. Let $a_j = \frac{1}{j!}$. Then

$$\sum_{j\in\mathbb{N}}\frac{1}{j!}\mathrm{e}^{xj\imath} = \mathrm{e}^{\mathrm{e}^{x\imath}} - 1 = \mathrm{e}^{\cos x + \imath\sin x} - 1$$
$$= \mathrm{e}^{\cos x}[\cos\sin x + \imath\sin\sin x] - 1.$$

So by (14) we derive

$$\Phi(x) = \frac{2}{x}\left[\sum_{j\in\mathbb{N}}\frac{1}{j!} - \mathrm{e}^{\cos x}\cos\sin x + 1\right] = \frac{2}{x}[\mathrm{e} - \mathrm{e}^{\cos x}\cos\sin x].$$

By Remark 1, $\Phi \in C^1(\mathbb{R}, \mathbb{R})$ with the graph on $[-4\pi, 4\pi]$:



a numerical solution shows that Φ has a maximum $\bar{R} = \Phi(x_0) \doteq 3.15177$ at $x_0 \doteq 1.03665$.

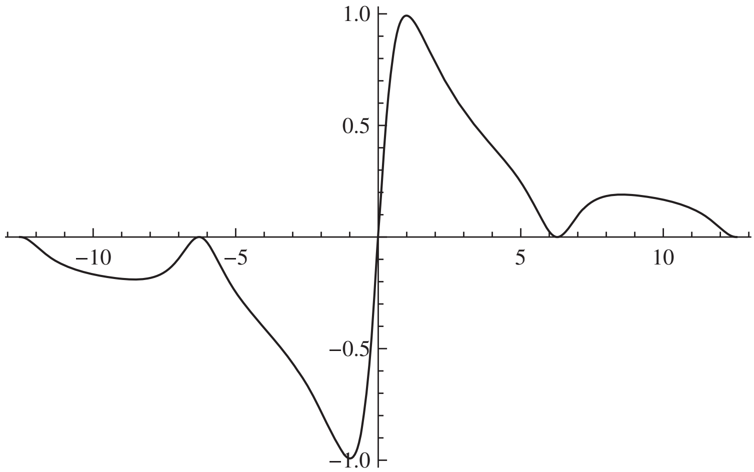
Example 2.7 Now we suppose that a_j is decaying exponentially to 0. Let $a_j = \mathrm{e}^{-j}$, hence we have the discrete Kac–Baker interaction kernel [20,21]. Then

$$\sum_{j\in\mathbb{N}}\mathrm{e}^{-j}\mathrm{e}^{xj\imath} = \sum_{j\in\mathbb{N}}\mathrm{e}^{(x\imath-1)j} = \frac{\mathrm{e}^{x\imath-1}}{1-\mathrm{e}^{x\imath-1}}$$
$$= \frac{\cos x + \imath\sin x}{\mathrm{e} - \cos x - \imath\sin x} = \frac{\mathrm{e}\cos x - 1 + \mathrm{e}\imath\sin x}{\mathrm{e}^2 + 1 - 2\,\mathrm{e}\cos x}.$$

So by (14) we derive

$$\Phi(x) = \frac{2}{x}\left[\sum_{j\in\mathbb{N}}\mathrm{e}^{-j} - \frac{\mathrm{e}\cos x - 1}{\mathrm{e}^2 + 1 - 2\,\mathrm{e}\cos x}\right] = \frac{2\,\mathrm{e}(\mathrm{e}+1)(1-\cos x)}{(\mathrm{e}-1)x(\mathrm{e}^2+1-2\,\mathrm{e}\cos x)}.$$

By Remark 1, $\Phi \in C^1(\mathbb{R}, \mathbb{R})$ with the graph on $[-4\pi, 4\pi]$:



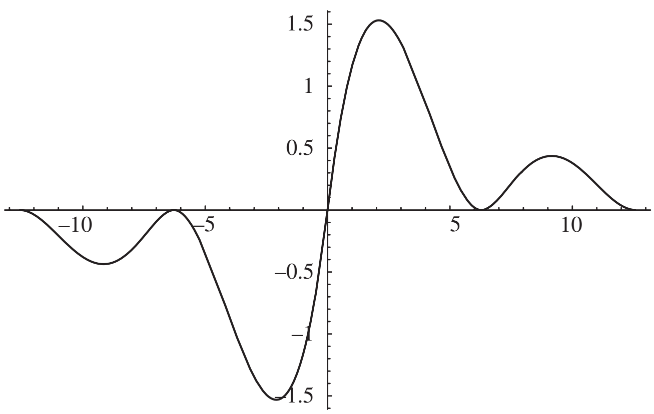
A numerical solution shows that Φ has a maximum $\bar{R} = \Phi(x_0) \doteq 0.992045$ at $x_0 \doteq 0.991541$.

Example 2.8 In this example, we suppose that a_j is decaying polynomially to 0 (cf [23]), by considering several cases

(1) Let $a_j = \frac{1}{j4}$. Then

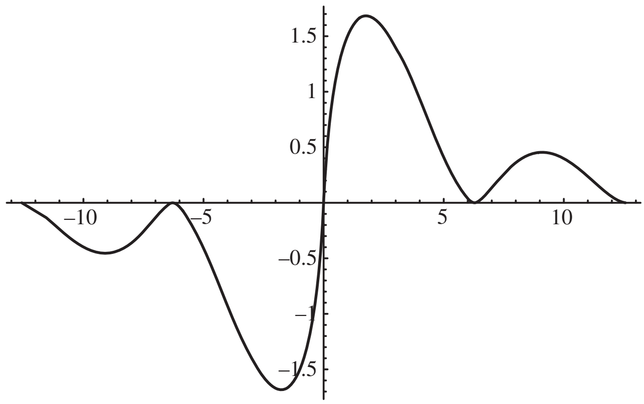
$$\Phi(x) = \frac{2}{x}\sum_{j\in\mathbb{N}}\left(\frac{1}{j^4} - \frac{1}{j4}\cos xj\right) = \frac{\left(|x| - 2\pi\left[\frac{|x|}{2\pi}\right]\right)^2}{24x}\left(2\pi - |x| + 2\pi\left[\frac{|x|}{2\pi}\right]\right)^2.$$

Here $[\cdot]$ is the integer part function. By Remark 1, $\Phi \in C^1(\mathbb{R}, \mathbb{R})$ with the graph on $[-4\pi, 4\pi]$:



Φ has a maximum $\bar{R} = \Phi(x_0) = \frac{4\pi^3}{81} \doteq 1.53117$ at $x_0 = 2\pi/3 \doteq 2.0944$. Similar results hold for $a_j = j^{-\beta}$ with $\beta > 3$ by Remark 1.

(2) Let $a_j = \frac{1}{j^3}$. So we consider the dipole–dipole interaction (cf [16,20,22,23]). By Remark 1, $\Phi \in C(\mathbb{R}, \mathbb{R})$ with the graph on $[-4\pi, 4\pi]$:



Φ has a maximum $\bar{R} = \Phi(x_0) = \frac{4\pi^3}{81} \doteq 1.68311$ at $x_0 \doteq 1.76076$. Next we know that [40]

$$\sum_{j\in\mathbb{N}} \frac{1}{j} \cos xj = -\ln\left|2\sin\frac{x}{2}\right|, \quad 0 < x < 2\pi.$$

Then

$$\sum_{j\in\mathbb{N}} \frac{1}{j^2} \sin xj = -\int_0^x \ln\left|2\sin\frac{s}{2}\right| \mathrm{d}s.$$

Using $x/2 \le \sin x \le x$ for $x \ge 0$ small, we derive

$$x - x\ln x = -\int_0^x \ln s\, \mathrm{d}s \le \sum_{j\in\mathbb{N}} \frac{1}{j^2} \sin xj \le -\int_0^x \ln\frac{s}{2} \mathrm{d}s = x - x\ln\frac{x}{2}.$$

By L'Hopital's rule, we obtain

$$\lim_{x\to 0_+} \frac{\Phi(x)}{x} = \lim_{x\to 0_+} \frac{4\sum_{j\in\mathbb{N}} \frac{1}{j^3}\sin^2 xj}{x^2} = \lim_{x\to 0_+} \frac{2\sum_{j\in\mathbb{N}} \frac{1}{j^2}\sin 2xj}{x} = +\infty.$$

Hence Φ has no derivative at $x_0 = 0$.

Next, let $a_j = j^{-\beta}$ for $2 < \beta < 3$. By Remark 1, Φ is still continuous. Since $\Phi(0) = 0$ and

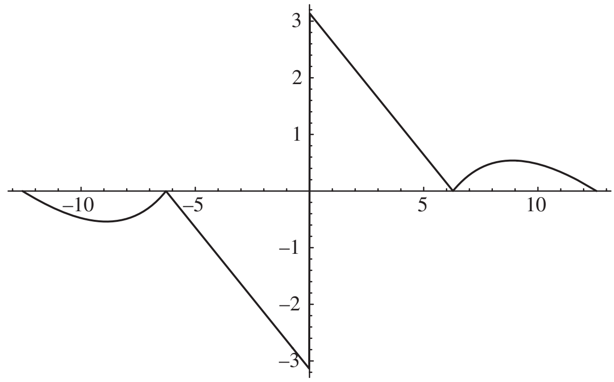
$$\lim_{x\to 0_+} \frac{\Phi(x)}{x} \ge \lim_{x\to 0_+} \frac{4\sum_{j\in\mathbb{N}} \frac{1}{j^3}\sin^2 xj}{x^2} = +\infty,$$

$\Phi(x)$ is continuous but not C^1-smooth on $\mathbb{R}$.

(3) Let $a_j = \frac{1}{j^2}$. Then

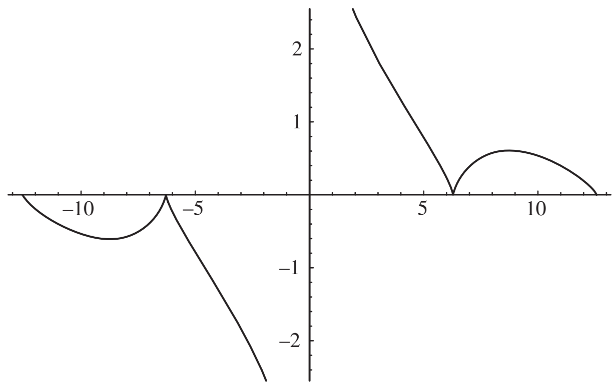
$$\Phi(x) = \frac{2}{x}\sum_{j\in\mathbb{N}}\left(\frac{1}{j^2} - \frac{1}{j^2}\cos xj\right) = \frac{\left(|x| - 2\pi\left[\frac{|x|}{2\pi}\right]\right)}{2x}\left(2\pi - |x| + 2\pi\left[\frac{|x|}{2\pi}\right]\right).$$

By Remark 1, $\Phi \in C(\mathbb{R} \setminus \{0\}, \mathbb{R})$ with the graph on $[-4\pi, 4\pi]$:



Φ is discontinuous at $x_0 = 0$ where it has a supremum $\bar{R} = \pi$.

(4) Let $a_j = j^{-\beta}$ for $1 < \beta < 2$. For $\beta = 7/4$, Φ has the graph on $[-4\pi, 4\pi]$:



Hence Φ is discontinuous at $x_0 = 0$ with $\lim_{x \to 0_+} \Phi(x) = +\infty$. We show that this holds for any $1 < \beta < 2$. First suppose $3/2 < \beta < 2$. Then the series

$$\Upsilon(x) := \sum_{j \in \mathbb{N}} \frac{1}{j^{\beta-1}} \sin jx$$

converges uniformly on any $[\varepsilon, 2\pi - \varepsilon]$ for $0 < \varepsilon < \pi$. But since $\sum_{j \in \mathbb{N}} \frac{1}{j^{2(\beta-1)}} < \infty$, so $\Upsilon \in L^2 \subset L^1$. On the other hand, we know [40] that

$$\Upsilon(x) := \Gamma(2-\beta) \cos \frac{\pi(\beta-1)}{2} \cdot x^{\beta-2} + O(1)$$

on $(0, \pi]$. Hence

$$\sum_{j \in \mathbb{N}} \frac{1 - \cos jx}{j^\beta} = \int_0^x \Upsilon(s) \mathrm{d}s = \frac{\Gamma(2-\beta)}{\beta - 1} \cos \frac{\pi(\beta-1)}{2} \cdot x^{\beta-1} + O(x)$$

on $[0, \pi]$. Consequently, we obtain

$$\Phi(x) = \frac{2\Gamma(2-\beta)}{\beta-1} \cos \frac{\pi(\beta-1)}{2} \cdot x^{\beta-2} + O(1)$$

on $(0,\pi]$, which implies $\lim_{x\to 0_+}\Phi(x)=+\infty$ for any $3/2<\beta<2$. Finally, if $1<\beta\le 3/2$, then

$$\Phi(x)\ge \frac{2}{x}\sum_{j\in\mathbb{N}}\frac{1-\cos jx}{j^{7/4}}=\frac{8}{3}\Gamma\left(\frac{1}{4}\right)\cos\frac{3\pi}{8}\cdot\frac{1}{\sqrt[4]{x}}+O(1)\to+\infty$$

as $x\to 0_+$. Hence, $\lim_{x\to 0_+}\Phi(x)=+\infty$ for any $1<\beta<2$.

Summarizing, we have the following result.

LEMMA 2.9 *Let $a_j=j^{-\beta}$ for $1<\beta$. Then*

(i) $\Phi\in C^1(\mathbb{R},\mathbb{R})$ *for* $\beta>3$, *and* $\mathcal{R}\Phi=[-\bar{R},\bar{R}]$ *for some* $\bar{R}<\infty$.
(ii) $\Phi\in C(\mathbb{R},\mathbb{R})$ *and* $\Phi\notin C^1(\mathbb{R},\mathbb{R})$ *for* $2<\beta\le 3$, *and* $\mathcal{R}\Phi=[-\bar{R},\bar{R}]$ *for some* $\bar{R}<\infty$.
(iii) $\Phi\in C(\mathbb{R}\setminus\{0\},\mathbb{R})$ *and* $\Phi\notin C(\mathbb{R},\mathbb{R})$ *for* $\beta=2$, *and* $\mathcal{R}\Phi=(-\pi,\pi)$.
(iv) $\Phi\in C(\mathbb{R}\setminus\{0\},\mathbb{R})$ *and* $\Phi\notin C(\mathbb{R},\mathbb{R})$ *for* $1<\beta<2$, *and* $\mathcal{R}\Phi=(-\infty,+\infty)$.

Remark 5 We see that if the interaction is strong, so the case (iv) of Lemma 2.9 holds, then there are continuum many quasi periodic travelling wave solutions $U(z)$ of Theorem 1.1 for any $\nu\neq 0$, $T>0$ and $r\in(0,1)$ such that $r\notin\{z-[z]\mid z\in\frac{T}{2\pi}\Phi^{-1}(-\nu)\}$, with $\|U\|_{Y_r}\to\infty$ as $|\nu|\to\infty$ by Remark 4. On the other hand, if the interaction is weak, then we can show in addition quasi periodic travelling waves with speeds in intervals $(-\infty,-\bar{R})$ and $(\bar{R},\infty)$ for any $T>0$ and $r\in(0,1)$.

Remark 6 For the reader convenience, we present the above graphs of function Φ to visualize their quantitative and qualitative changes according to different choices of values of sequences $\{a_j\}_{j\in\mathbb{Z}}$ in (3), and hence with different consequences from Theorems 1.1 and 1.2 for the existence and bifurcations of quasi periodic travelling wave solutions of (3). Moreover, these graphs can be compared with similar ones for travelling waves for higher dimensional DNLS in Section 4 and for travelling waves with frequencies in Section 5. Finally these examples are motivated by applications mentioned in the corresponding references.

3. Bifurcation of travelling wave solutions

In this section we proceed with the study of (6) when nonresonance of Theorem 1.1 fails, i.e. $r\in\{\bar{r}_1,\bar{r}_2,\ldots,\bar{r}_m\}$. We scale in (6) the velocity by $\nu\leftrightarrow\nu/(1+\lambda)$ to get equation

$$-\nu\imath U'(z)=(1+\lambda)\left(\sum_{j\in\mathbb{N}}a_j\partial_j U(z)+f(|U(z)|^2)U(z)\right), \tag{15}$$

where λ is a small parameter, i.e. $u_n(t)=U\left(n-\frac{\nu}{1+\lambda}t\right)$ is a solution of (3). We are interested in the existence of quasi periodic solutions $U(z)$ of (15) stated in Theorem 1.2.

3.1. *Preliminaries*

In this subsection we recall some results from critical point theory of [41]. Let H be a Hilbert space with a scalar product $(\cdot, \cdot)$ and the corresponding norm $\|\cdot\|$. Let $\Theta : S^1 \to L(H)$ be an isometric representation of the unit circle S^1 over H, i.e. the following properties are satisfied

(R) $\Theta(0) = \mathrm{I}$ – the identity, $\Theta(\theta_1 + \theta_2) = \Theta(\theta_1)\Theta(\theta_2)$ for any $\theta_1, \theta_2 \in S^1$, $(\theta, h) \to \Theta(\theta)h$ is continuous and $\|\Theta(\theta)h\| = \|h\|$ for any $\theta \in S^1$ and $h \in H$.

We set

$$\mathrm{Fix}(S^1) := \left\{ h \in H \mid \Theta(\theta)h = h \quad \forall \theta \in \Theta \right\}.$$

We consider $J_1, J_2 \in C^2(H, \mathbb{R})$ such that

(H1) $J_2(0) = 0$ and $\nabla J_1(0) = \nabla J_2(0) = 0$.
(H2) Hess $J_1(0)$ is a Fredholm operator, i.e. dim Hess $J_1(0) < \infty$, $\mathcal{R}$Hess $J_1(0)$ is closed and codim $\mathcal{R}$Hess $J_1(0) < \infty$.
(H3) dim ker Hess $J_1(0) \geq 2$ and Hess $J_2(0)$ is positive definite on ker Hess $J_1(0)$.
(H4) J_1 and J_2 are S^1-invariant, i.e. $J_{1,2}(\Theta(\theta)h) = \Theta(\theta)J_{1,2}(h)$ for any $\theta \in \Theta$ and $h \in H$.
(H5) ker Hess $J_1(0) \cap \mathrm{Fix}(S^1) = \{0\}$.

Now we can state the following [41, Theorem 6.7].

THEOREM 3.1 *Under the above assumptions* (H1)–(H5), *for each sufficiently small* $\varepsilon > 0$, *equation*

$$\nabla J_1(h) + \lambda \nabla J_2(h) = 0 \tag{16}$$

has at least $\frac{1}{2}\dim\ker\mathrm{Hess}\, J_1(0)$ *of* S^1-*orbit solutions*

$$\left\{ (\lambda_k(\varepsilon), \Theta(\theta)h_k(\varepsilon)) \mid \theta \in S^1 \right\}, \quad k = 1, 2, \ldots, \tfrac{1}{2}\dim\ker\mathrm{Hess}\, J_1(0)$$

such that $J_2(h_k(\varepsilon)) = \varepsilon$ *and* $h_k(\varepsilon) \to 0$, $\lambda_k(\varepsilon) \to 0$ *as* $\varepsilon \to 0$. Clearly $h_k(\varepsilon) \neq 0$.

Remark 1 When Hess $J_2(0)$ is negative definite on ker Hess $J_1(0)$, then Theorem 3.1 holds for $\varepsilon < 0$ small.

Remark 2 By (H4), ker Hess $J_1(0)$ is invariant with respect to Θ. Using (H5), dim ker Hess $J_1(0)$ is even.

Now assume $H = H_+ \oplus H_-$ be an orthogonal and Θ-invariant decomposition with the corresponding orthogonal projections $P_\pm : H \to H_\pm$. Then $\Theta(\theta)P_\pm = P_\pm\Theta(\theta)$ for any $\theta \in \Theta$. Let us consider the equation

$$\zeta(\boldsymbol{I}_+ - \boldsymbol{I}_-)h + (1 + \lambda)(\mathcal{K}h + \nabla\mathcal{F}(h)) = 0, \tag{17}$$

where $\zeta \neq 0$ is a constant, λ is a small parameter, $\boldsymbol{I}_\pm : H_\pm \to H_\pm$ are the identities. We suppose

(A) $\mathcal{K} : H \to H$ is compact self-adjoint and $\mathcal{F} \in C^2(H, \mathbb{R})$ with $\mathcal{F}(0) = 0$, $\nabla\mathcal{F}(0) = 0$, Hess $\mathcal{F}(0) = 0$ and $\mathcal{K}$, $\mathcal{F}$ are S^1-invariant. Moreover, $\mathcal{K}H_\pm \subset H_\pm$.

Then

$$J_1(h) = \frac{\zeta}{2}(\|P_+h\|^2 - \|P_-h\|^2) + \frac{1}{2}(\mathcal{K}h, h) + \mathcal{F}(h),$$
$$J_2(h) = \tfrac{1}{2}(\mathcal{K}h, h) + \mathcal{F}(h).$$

Hence

$$J_1(0) = J_2(0) = 0, \quad \nabla J_1(0) = \nabla J_2(0) = 0,$$
$$\text{Hess}\, J_1(0) = \zeta(\boldsymbol{I}_+ - \boldsymbol{I}_-) + \mathcal{K}, \quad \text{Hess}\ J_2(0) = \mathcal{K}.$$

So assumptions (H1), (H2) and (H4) are satisfied. Since $P_\pm\mathcal{K} = \mathcal{K}P_\pm$, equation

$$\text{Hess}\, J_1(0)h = \zeta(\boldsymbol{I}_+ - \boldsymbol{I}_-)h + \mathcal{K}h = 0$$

splits into

$$\mathcal{K}h_+ = -\zeta h_+, \quad \mathcal{K}h_- = \zeta h_-, \quad h_\pm = P_\pm h.$$

Consequently, supposing either

(B_+) $\ker(\zeta\boldsymbol{I} + \mathcal{K}) \cap H_+ = \{0\}$, $\dim\ker(\zeta\boldsymbol{I} - \mathcal{K}) \cap H_- \geq 2$ and $\ker(\zeta\boldsymbol{I} - \mathcal{K}) \cap H_- \cap \text{Fix}(S^1) = \{0\}$

or

(B_-) $\ker(\zeta\boldsymbol{I} - \mathcal{K}) \cap H_- = \{0\}$, $\dim\ker(\zeta\boldsymbol{I} + \mathcal{K}) \cap H_+ \geq 2$ and $\ker(\zeta\boldsymbol{I} + \mathcal{K}) \cap H_+ \cap \text{Fix}(S^1) = \{0\}$

we get either

$$\ker \text{Hess}\, J_1(0) = \ker(\zeta\boldsymbol{I} - \mathcal{K}) \cap H_-$$

or

$$\ker \text{Hess}\, J_1(0) = \ker(\zeta\boldsymbol{I} + \mathcal{K}) \cap H_+$$

and so (H5) holds as well. Finally, we derive

$$\text{Hess}\, J_2(0) |\ker \text{Hess}\, J_1(0) = \pm\zeta\boldsymbol{I}$$

and thus (H3) is also verified (cf Remark 1). Summarizing, Theorem 3.1 and Remark 1 is applicable to (17):

Corollary 3.2 *Under assumptions* (A) *and* ($B_\pm$), *for each sufficiently small* $\varepsilon \neq 0$, $\pm\varepsilon\zeta > 0$, *Equation* (17) *has at least* $\frac{1}{2}\dim\ker(\zeta\boldsymbol{I} \mp \mathcal{K}) \cap H_\mp$ *of* S^1*-orbit solutions*

$$\left\{(\lambda_k(\varepsilon), \Theta(\theta)h_k(\varepsilon)) \mid \theta \in S^1\right\}, \quad k = 1, 2, \ldots, \tfrac{1}{2}\dim\ker(\zeta\boldsymbol{I} \mp \mathcal{K}) \cap H_\mp$$

such that $\frac{1}{2}(\mathcal{K}h_k(\varepsilon), h_k(\varepsilon)) + \mathcal{F}(h_k(\varepsilon)) = \varepsilon$ *and* $h_k(\varepsilon) \to 0$, $\lambda_k(\varepsilon) \to 0$ *as* $\varepsilon \to 0$. *Clearly* $h_k(\varepsilon) \neq 0$.

Remark 3 If $\text{Fix}(S^1) = \{0\}$ then (B_+) holds if

(i) $-\zeta \notin \sigma(\mathcal{K}/H_+)$, $\zeta \in \sigma(\mathcal{K}/H_-)$ and ζ has a multiplicity at least 2, while (B_-) holds if
(ii) $-\zeta \in \sigma(\mathcal{K}/H_+)$, $\zeta \notin \sigma(\mathcal{K}/H_-)$ and $-\zeta$ has a multiplicity at least 2,

respectively.

3.2. *Proof of Theorem 1.2*

We again assume for simplicity $T=2\pi$. So let $r=\bar{r}_1\in(0,1)$ and the equation

$$-\nu=\Phi(\bar{r}_1+k)$$

has solutions $k_1,k_2,\dots,k_{m_1}\in\mathbb{Z}$ which are either all nonnegative or all negative. Next (15) has the form (cf (9))

$$2(\nu\boldsymbol{I}_+-\nu\boldsymbol{I}_-)-(1+\lambda)\Big(\widetilde{K}_r\mathcal{L}_rU+\Psi_r(U)\Big)=0 \tag{18}$$

and

$$H=X_r,\quad \zeta=2\nu,\quad H_\pm=X_\pm,$$
$$\mathcal{K}=-\widetilde{K}_r\mathcal{L}_r,\quad \mathcal{F}(u)=-\int_0^{2\pi}F(|U(z)|^2)\mathrm{d}z.$$

Isometric representation Θ is naturally given as

$$\Theta(\theta)U(z):=U(z+\theta),$$

i.e.

$$\Theta(\theta)\left(\sum_{k\in\mathbb{Z}}U_k\,\mathrm{e}^{(\bar{r}_1+k)z\imath}\right)=\sum_{k\in\mathbb{Z}}U_k\,\mathrm{e}^{\theta k\imath}\,\mathrm{e}^{(\bar{r}_1+k)z\imath}.$$

Note $\mathrm{Fix}(S^1)=\{0\}$. It is easy to verify (R) for Θ. By the results of Section 2, we get both $\mathcal{K}H_\pm\subset H_\pm$ and assumption (A) holds and moreover

$$\sigma(\mathcal{K}/H_\pm)=\big\{\pm2\Phi(\bar{r}_1+k)\mid k\in\mathbb{Z}_\pm\big\}.$$

Note $\mathbb{Z}_+=\{0\}\cup\mathbb{N}$ and $\mathbb{Z}_-=-\mathbb{N}$. Hence (i) of Remark 3 is satisfied if

$$-\nu\notin\{\Phi(\bar{r}_1+k)\mid k\in\mathbb{Z}_+\},\quad -\nu\in\big\{\Phi(\bar{r}_1+k)\mid k\in\mathbb{Z}_-\big\},$$

while (ii) if

$$-\nu\in\big\{\Phi(\bar{r}_1+k)\mid k\in\mathbb{Z}_+\big\},\quad -\nu\notin\{\Phi(\bar{r}_1+k)\mid k\in\mathbb{Z}_-\}.$$

But these are precisely assumptions of Theorem 1.2. So its proof is complete by Corollary 3.2 and Remark 3.

4. Travelling waves for higher dimensional DNLS

In this section, we first show how to extend previous results for 2-dimensional DNLS (2D DNLS) [24–26] of forms

$$\begin{aligned}\imath\dot{u}_{n,m}&=\sum_{(i,j)\in\mathbb{Z}_0^2}a_{i,j}\Delta_{i,j}u_{n,m}+f(|u_{n,m}|^2)u_{n,m},\quad (n,m)\in\mathbb{Z}^2\\ &=2\sum_{(i,j)\in\mathbb{Z}_0^2}a_{i,j}\big(u_{n+i,m+j}-u_{n,m}\big)+f(|u_{n,m}|^2)u_{n,m},\end{aligned} \tag{19}$$

where $u_{n,m}\in\mathbb{C}$, $\mathbb{Z}_0^2:=\mathbb{Z}^2\setminus\{(0,0)\}$, $\Delta_{i,j}u_{n,m}:=u_{n+i,m+j}+u_{n-i,m-j}-2u_{n,m}$ are 2-dimensional discrete Laplacians, f satisfies (H1) and $a_{i,j}=a_{-i,-j}$ along with $\sum_{(i,j)\in\mathbb{Z}_0^2}|a_{i,j}|<\infty$ and all $a_{i,j}$ are not zero.

Again, (19) conserves two dynamical invariants

$$\sum_{(n,m)\in\mathbb{Z}^2} |u_{n,m}|^2 - \text{the norm},$$

$$\sum_{(n,m)\in\mathbb{Z}^2}\left[-\sum_{(i,j)\in\mathbb{Z}_0^2} a_{i,j}\left|u_{n+i,m+j}-u_{n,m}\right|^2 + F(|u_{n,m}|^2)\right] - \text{the energy}.$$

We look for travelling wave solutions of (19) of the form

$$u_{n,m}(t) = U(n\cos\theta + m\sin\theta - \nu t) \tag{20}$$

with a direction $(\cos\theta, \sin\theta)$ [39]. Hence we are interested in the equation

$$-\nu\imath U'(z) = \sum_{(i,j)\in\mathbb{Z}_0^2} a_{i,j}\partial_{i,j}U(z) + f(|U(z)|^2)U(z), \tag{21}$$

where $z = n\cos\theta + m\sin\theta - \nu t$, $\nu \neq 0$ and

$$\partial_{i,j}U(z) := U(z + i\cos\theta + j\sin\theta) + U(z - i\cos\theta - j\sin\theta) - 2U(z).$$

We see that (21) has a very similar form like (6). So we can directly repeat the above arguments, where now instead of $\Phi(x)$ we get

$$\Phi_\theta(x) := \frac{4}{x}\sum_{(i,j)\in\mathbb{Z}_0^2} a_{i,j}\sin^2\frac{x(i\cos\theta + j\sin\theta)}{2}.$$

Set $\bar{R}_\theta := \sup_{\mathbb{R}}\Phi_\theta$. Summarizing, Theorems 1.1 and 1.2 have the following analogies.

THEOREM 4.1 *Let* (H1) *hold and* $T>0$, $\theta\in[0, 2\pi)$. *Then for almost each* $\nu\in\mathbb{R}\setminus\{0\}$ *and any rational* $r\in\mathbb{Q}\cap(0,1)$, *there is a nonzero periodic travelling wave solution* (20) *of* (19) *with* $U\in C^1(\mathbb{R},\mathbb{C})$ *satisfying* (5). *Moreover, for any* $\nu\in\mathbb{R}\setminus\{0\}$, *there is at most a finite number of* $\bar{r}_{1,\theta}, \bar{r}_{2,\theta}, \ldots, \bar{r}_{m_\theta,\theta}\in(0,1)$ *such that equation*

$$-\nu = \Phi_\theta\left(\frac{2\pi}{T}(\bar{r}_{j,\theta} + k)\right)$$

has a solution $k\in\mathbb{Z}$. *Then for any* $r\in(0,1)\setminus\{\bar{r}_{1,\theta}, \bar{r}_{2,\theta}, \ldots, \bar{r}_{m_\theta,\theta}\}$ *there is a nonzero quasi periodic travelling wave solution* (20) *of* (19) *with the above properties. In particular, for any* $|\nu| > \bar{R}_\theta$ *and* $r\in(0,1)$, *and* $r\in(0,1)$, *there is such a nonzero quasi periodic travelling wave solution.*

THEOREM 4.2 *Suppose* $f\in C^1(\mathbb{R}_+, \mathbb{R})$ *with* $f(0)=0$. *If there are* $\bar{r}_{1,\theta}\in(0,1)$, $T>0$, $\theta\in[0, 2\pi)$ *and* $\nu\in\mathcal{R}\Phi_\theta\setminus\{0\}$ *such that all integer number solutions* $k_1, k_2, \ldots, k_{m_1,\,\theta}$ *of equation*

$$-\nu = \Phi_\theta\left(\frac{2\pi}{T}(\bar{r}_{1,\theta} + k)\right)$$

are either nonnegative or negative and $m_{1,\theta}>0$. *Then for any* $\varepsilon>0$ *small there are* $m_{1,\theta}$ *branches of nonzero quasi periodic travelling wave solutions* (20) *of* (19) *with* $U_{j,\varepsilon}\in C^1(\mathbb{R},\mathbb{C})$, $j=1,2,\ldots,m_{1,\theta}$ *and nonzero velocity* ν_ε *satisfying* $U_{j,\varepsilon}(z+T) = \mathrm{e}^{2\pi\bar{r}_1\imath}U(z)_{j,\varepsilon}$, $\forall z\in\mathbb{R}$ *along with* $\nu_\varepsilon\to\nu$ *and* $U_{j,\varepsilon}\rightrightarrows 0$ *uniformly on* $\mathbb{R}$ *as* $\varepsilon\to 0$.

Example 4.3 We consider the discrete 2D Kac–Baker interaction kernel $a_{i,j} = \mathrm{e}^{-|i|-|j|}$ for $(i, j) \in \mathbb{Z}^2{}_0$. Then $\sum_{(i,j)\in\mathbb{Z}_0^2} \mathrm{e}^{-|i|-|j|} = \frac{4\,\mathrm{e}}{(e-1)^2}$ and

$$\Phi_\theta(x) = \left[\frac{(\mathrm{e}+1)^2}{(\mathrm{e}-1)^2} - \frac{(\mathrm{e}^2-1)^2}{(1+\mathrm{e}^2-2\,\mathrm{e}\cos(x\cos\theta))(1+\mathrm{e}^2-2\,\mathrm{e}\cos(x\sin\theta))}\right]\frac{4}{x}.$$

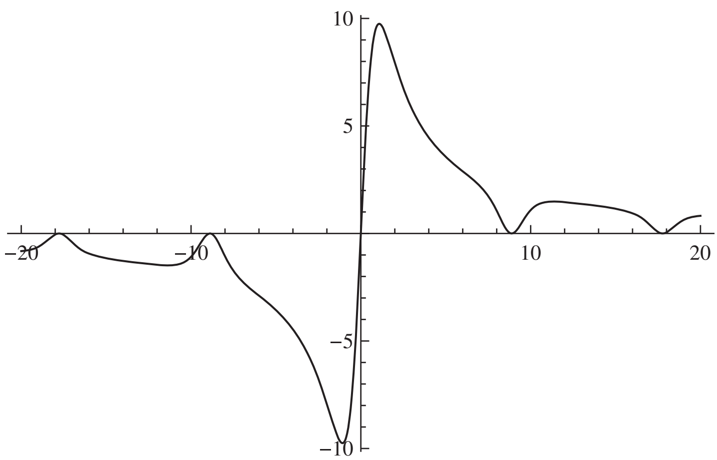
A numerical evaluation shows that function $(x,\theta)\to\Phi_\theta(x)$ has a maximum $\bar{R} \doteq 9.75047$ at $x_0 \doteq 1.08205$ and $\theta_0 \doteq 0.785398$. To justify this theoretically, we take $a = x\cos\theta$ and $b = x\sin\theta$ to transform $\Phi_\theta(x)$ into

$$\Phi(a,b) = \left[\frac{(\mathrm{e}+1)^2}{(\mathrm{e}-1)^2} - \frac{(\mathrm{e}^2-1)^2}{(1+\mathrm{e}^2-2\,\mathrm{e}\cos a)(1+\mathrm{e}^2-2\,\mathrm{e}\cos b)}\right]\frac{4}{\sqrt{a^2+b^2}}.$$

Note $\Phi(a, b) = \Phi(\pm a, \pm b) = \Phi(b, a)$. A numerical evaluation shows that function $\Phi(a, b)$ has a maximum $\bar{R} \doteq 9.75047$ at $a_0 = b_0 \doteq 0.765123$ which correspond to x_0 and θ_0. On the other hand, if $a^2+b^2 \geq 4$ then $\Phi(a,b) \leq 2\frac{(\mathrm{e}+1)^2}{(\mathrm{e}-1)^2} \doteq 9.36539 < 9.75047$, so $\Phi(a, b)$ achieves its maximum in the disc $D_2 := \{a^2+b^2 \leq 4\}$. Next, solving the system $\frac{\partial}{\partial a}\Phi(a,b) = \frac{\partial}{\partial b}\Phi(a,b) = 0$ we derive $\frac{\sin a_0}{a_0} = \frac{\sin b_0}{b_0}$ at the maximum point $(a_0, b_0) \in D_2$, $a_0 > 0$, $b_0 > 0$. But the function $\frac{\sin w}{w}$ is decreasing on [0, 2], so $a_0 = b_0$ and thus $\theta_0 = \pi/4$. An elementary but awkward calculus shows for function

$$\Phi_{\pi/4}(x) = \left[\frac{(\mathrm{e}+1)^2}{(\mathrm{e}-1)^2} - \frac{(\mathrm{e}^2-1)^2}{\left(1+\mathrm{e}^2-2\,\mathrm{e}\cos\left(x\frac{\sqrt{2}}{2}\right)\right)^2}\right]\frac{4}{x}$$

with the graph on [−20, 20]:



that $x_0 \in (0, 2)$ is the only root of $\Phi'_{\pi/4}(x_0) = 0$ on $(0, 2)$ and then $\bar{R} = \Phi_{\pi/4}(x_0)$. So $\bar{R}$ is computed also analytically in this case.

Summarizing, Theorems 4.1 and 4.2 can be applied in this case for any suitable nonzero ν and resonant travelling waves with maximum velocities which are achieved in the diagonal directions $\pm\theta_0 = \pm\pi/4$.

Finally, it is now clear how to proceed to 3D DNLS or even to higher dimensional DNLS, so we omit further details.

5. Travelling waves with frequencies

We could consider more general travelling wave solutions than the above forms

$$\begin{aligned} u_n(t) &= U(n-\nu t)\,\mathrm{e}^{\imath\omega t},\\ u_{n,m}(t) &= U(n\cos\theta + m\sin\theta - \nu t)\,\mathrm{e}^{\imath\omega t} \end{aligned} \tag{22}$$

with velocity $\nu \neq 0$ and frequency $\omega \neq 0$ [29]. Then, there is a dispersion relation between the velocity ν and frequency ω as follows. Inserting (22) into (3) and (19), respectively, we are interested in equations

$$\begin{aligned} -\nu\imath U'(z) &= \sum_{j\in\mathbb{N}} a_j\partial_j U(z) + \omega U(z) + f(|U(z)|^2)U(z),\\ -\nu\imath U'(z) &= \sum_{(i,j)\in\mathbb{Z}_0^2} a_{i,j}\partial_{i,j} U(z) + \omega U(z) + f(|U(z)|^2)U(z), \end{aligned} \tag{23}$$

respectively. We see that (6), (21) and (23) are very similar, so we can repeat the above arguments to (23) when instead of $\Phi(x)$ and $\Phi_\theta(x)$ now we have

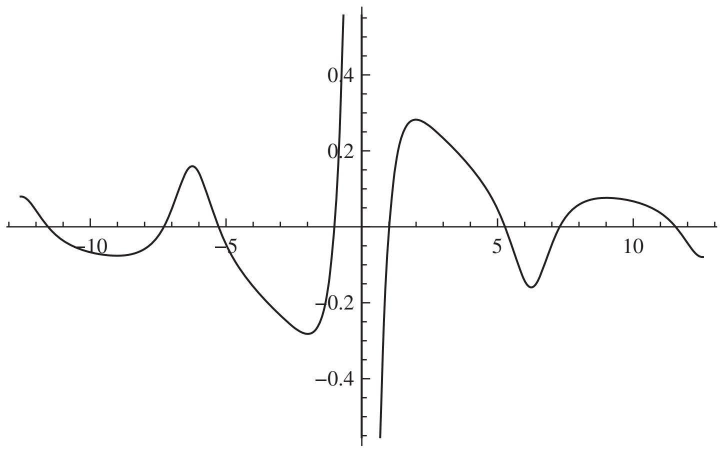
$$\Phi(x,\omega) := \Phi(x) - \frac{\omega}{x}, \quad \Phi_\theta(x,\omega) := \Phi_\theta(x) - \frac{\omega}{x}, \tag{24}$$

respectively. Consequently, we have analogies of Theorems 1.1, 1.2, 4.1 and 4.2 to (23) but we do not state them since they are obvious.

Example 5.1 We consider the discrete Kac–Baker interaction kernel from Example 2.7. Then

$$\Phi(x,\omega) = \frac{2\,\mathrm{e}(\mathrm{e}+1)(1-\cos x)}{(\mathrm{e}-1)x(\mathrm{e}^2+1-2\,\mathrm{e}\cos x)} - \frac{\omega}{x}.$$

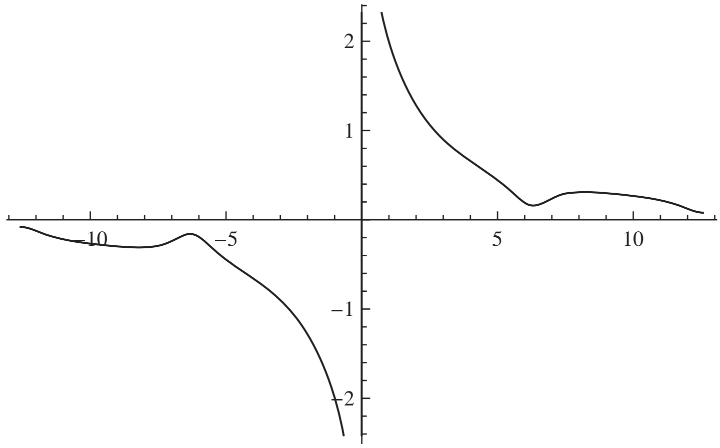
To be more concrete, we first take $\omega = 1$ and then $\Phi(x, 1)$ has the graph on $[-4\pi, 4\pi]$:



with $\lim_{x\to 0_\pm}\Phi(x,1) = \mp\infty$. A numerical evaluation shows that function $\Phi(x, 1)$ has a maximum $\bar{R} \doteq 0.282071$ on $(0, \infty)$ at $x_0 \doteq 1.9905$. Consequently, the analogy of

Theorem 1.1 can be applied now to any $\nu \neq 0$ while the analogy of Theorem 1.1 can be applied for almost any $\nu \in \mathbb{R} \setminus [-0.282071, 0.282071]$, while for nonzero $\nu \in [-0.282071, 0.282071]$ could be problematic in general.

On the other hand for $\omega = -1$, $\Phi(x,-1)$ has the graph on $[-4\pi, 4\pi]$:



with $\lim_{x \to 0_{\pm}} \Phi(x, -1) = \pm\infty$. Consequently, the analogy of Theorem 1.1 can again be applied now to any $\nu \neq 0$ while the analogy of Theorem 1.2 can now be applied for almost any $\nu \neq 0$. Of course, now we have totally different situations than in Example 2.7 for travelling waves without frequencies by comparing the above graphs with that one in Example 2.7.

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References

[1] R.K. Dodd, J.C. Eilbeck, J.D. Gibbon, and H.C. Morris, *Solitons and Nonlinear Wave Equations*, Academic Press, London, 1984.
[2] A. Hasegawa and Y. Kodama, *Optical Fibre Solitons*, Cambridge University Press, Cambridge, UK, 1994.
[3] C. Sulem and P.L. Sulem, *The Nonlinear Schrödinger Equation*, Springer-Verlag, New York, 1999.
[4] A.B. Aceves, G.G. Luther, C. De Angelis, A.M Rubenchik, and S.K. Turitsyn, *Energy localization in nonlinear fiber arrays: Collapse-effect compressor*, Phys. Rev. Lett. 75 (1995), pp. 73–76.
[5] J.C. Eilbeck, P.S. Lomsdahl, and A.C. Scott, *The discrete self-trapping equation*, Physica D 16 (1985), pp. 318–338.
[6] H. Eisenberg, Y. Silberberg, R. Morandotti, A. Boyd, and J. Aitchison, *Discrete spatial optical solitons in waveguide arrays*, Phys. Rev. Lett. 81 (1998), pp. 3383–3386.

[7] G. Lenz, I. Talanina, and C. Martijn de Sterke, *Bloch oscillations in an array of curved optical waveguides*, Phys. Rev. Lett. 83 (1999), pp. 963–966.
[8] A.V. Ustinov, T. Doderer, I.V. Vernik, N.F. Pedersen, R.P. Huebener, and V.A. Oboznov, *Experiments with solitons in annular Josephson junctions*, Physica D 68 (1994), pp. 41–44.
[9] W.P. Wu, J.R. Schrieffer, and A.J. Heeger, *Soliton excitations in polyacetylene*, Phys. Rev. B 22 (1980), pp. 2099–2111.
[10] S. Aubry, *Breathers in nonlinear lattices: Existence, linear stability and quantization*, Physica D 103 (1997), pp. 201–250.
[11] A.A. Aigner, A.R. Champneys, and V.M. Rothos, *A new barrier to the existence of moving kinks in Frenkel–Kontorova lattices*, Physica D 186 (2003), pp. 148–170.
[12] A.S. Davydov, *Theory of Molecular Excitons*, Plenum, New York, 1971.
[13] A.C. Scott, *Nonlinear Science: Emergence and Dynamics of Coherent Structures*, Oxford University Press, UK, 1999.
[14] F.Kh. Abdullaev, Yu.V. Bludov, S.V. Dmitriev, P.G. Kevrekidis, and V.V. Konotop, *Generalized neighbor-interaction models induced by nonlinear lattices*, Phys. Rev. E 77 (2008), p. 016604.
[15] M. Agrotis, S. Lafortune, and P.G. Kevrekidis, *On a discrete version of the Korteweg-de Vries equation*, Discrete Contin. Dyn. Sys., Suppl. Vol. (2005), pp. 22–29.
[16] J.F. Archilla, P.L. Christiansen, and Yu.B. Gaididei, *Interplay of nonlinearity and geometry in DNA-related, Klein–Gordon model with a long-range dipole–dipole interaction*, Phys. Rev. E 65 (2001), p. 016609.
[17] P.W. Bates and A. Chmaj, *A discrete convolution model for phase transitions*, Arch. Rational Mech. Anal. 150 (1999), pp. 281–305.
[18] P.W. Bates, X. Chen, and A. Chmaj, *Travelling waves of bistable dynamics on lattice*, SIAM J. Math. Anal. 35 (2003), pp. 520–546.
[19] P.W. Bates and C. Zhang, *Travelling pulses for the Klein–Gordon equation on a lattice or continuum with long-range interaction*, Disc. Cont. Dyn. Sys. 16 (2006), pp. 235–252.
[20] P.L. Christiansen, G.I. Gadidei, Yu.B. Gadidei, M. Johansson, S.F. Mingaleev, and K.O. Rasmussen, *Effects of noise and nonlocal interactions in nonlinear dynamics of molecular systems*, Funct. Materials 5 (1998), pp. 315–318.
[21] A. Comech, J. Cuevas, and P.G. Kevrekidis, *Discrete peakons*, Physica D 207 (2005), pp. 137–160.
[22] P.V. Larsen, P.L Christiansen, O. Bang, J.F.R. Archilla, and Yu.B. Gaididei, *Energy funneling in a bent chain of Morse oscillators with a long-range coupling*, Phys. Rev. E 69 (2004), p. 026603.
[23] S.F. Mingaleev, Yu.B. Gaididei, and F.G. Mertens, *Solitons in anharmonic chains with power-law long-range interactions*, Phys. Rev. E 58 (1998), pp. 3833–3842.
[24] R. Carretero-González, P.G. Kevrekidis, B.A. Malomed, and D.J. Frantzeskakis, *Three-dimensional nonlinear lattices: From oblique vortices and octupoles to discrete diamonds and vortex cubes*, Phys. Rev. Lett. 94 (2005), pp. 203901-1–203901-4.
[25] C. Chong, R. Carretero-González, B.A. Malomed, and P.G. Kevrekidis, *Multistable solitons in higher-dimensional cubic-quintic nonlinear Schrödinger lattices*, Physica D 238 (2009), pp. 126–136.
[26] J. Gómez-Gardeñes, L.M. Floría, and A.R. Bishop, *Discrete breathers in two-dimensional anisotropic nonlinear Schrödinger lattices*, Physica D 216 (2006), pp. 31–43.
[27] M. Lukas, D.E. Pelinovsky, and P.G. Kevrekidis, *Lyapunov–Schmidt reduction algorithm for three-dimensional discrete vortices*, Physica D 237 (2008), pp. 339–350.
[28] S. Guo, L.S.W. Lamb, and B.W. Rink, *Branching patterns of wave trains in the FPU lattice*, Nonlinearity 22 (2009), pp. 283–299.

[29] D.E. Pelinovsky, *Translationally invariant nonlinear Schrödinger lattices*, Nonlinearity 19 (2006), pp. 2695–2716.
[30] S.N. Chow, J. Mallet-Paret, and W. Shen, *Traveling waves in lattice dynamical systems*, J. Differential Equations 149 (1998), pp. 248–291.
[31] P. Coullet and D. Walgraef, *Spatial forcing of 2d wave patterns*, Europhys. Lett. 10 (1989), pp. 525–531.
[32] M. Fečkan and V.M. Rothos, *Kink-like periodic travelling waves for lattice equations with on-site and intersite potentials*, Dynamics of PDEs 2(4) (2005), pp. 357–370.
[33] A. Pankov and K. Pflüger, *Traveling waves in lattice dynamical systems*, Math. Methods Appl. Sci. 23 (2000), pp. 1223–1235.
[34] J. Pouget, M. Remoissenet, and J.M. Tamga, *Energy self-localization and gap local pulses in a two-dimensional nonlinear lattice*, Phys. Rev. B 47(22) (1993), pp. 14866–14874.
[35] D. Smets and M. Willem, *Solitary waves with prescribed speed on infinite lattices*, J. Funct. Anal. 149 (1997), pp. 266–275.
[36] J.M. Tamga, M. Remoissenet, and J. Pouget, *Breathing solitary waves in a sine-Gordon two dimensional lattice*, Phys. Rev. Letters 75 (1995), pp. 357–361.
[37] M. Fečkan and V.M. Rothos, *Traveling waves in Hamiltonian systems on 2d lattices with nearest neighbor interactions*, Nonlinearity 20 (2007), pp. 319–341.
[38] S. Li and A. Szulkin, *Periodic solutions for a class of nonautonomous Hamiltonian systems*, J. Differential Equations 112 (1994), pp. 226–238.
[39] P.H. Rabinowitz, *Minimax Methods in Critical Point Theory with Applications to Differential Equations*, CBMS Regional Conference Series in Mathematics, American Mathematical Society, Providence, RI, 1986.
[40] A. Zygmund, *Trigonometric Series*, 3rd ed., Cambridge University Press, Cambridge, UK, 2002.
[41] J. Mawhin and M. Willem, *Critical Point Theory and Hamiltonian Systems*, Springer, New York, 1989.