European Journal of Physics

PAPER • **OPEN ACCESS**

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To cite this article: John A Milsom 2023 *Eur. J. Phys.* **44** 055203

OPEN ACCESS
IOP Publishing

European Journal of Physics

Eur. J. Phys. **44** (2023) 055203 (8pp)

https://doi.org/10.1088/1361-6404/aceadf

Why are the electric and magnetic fields in an electromagnetic wave propagating through a conductor not in phase?

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Received 6 April 2023, revised 24 July 2023
Accepted for publication 26 July 2023
Published 24 August 2023

CrossMark

Abstract

Intermediate and advanced texts in electromagnetic theory frequently discuss infinite plane waves propagating through conducting media. They find that the magnetic field has a phase delay (relative to the electric field) that can be as large as $\pi/4$ rad depending upon the ratio of the conductivity to the product of the angular frequency and the permittivity $[\sigma/(\omega\epsilon)]$. The expressions given to calculate this phase delay are unnecessarily complicated and provide minimal physical insight. We provide a simple expression for the phase delay and then illustrate how to interpret it by first considering Ampere's Law and Faraday's Law separately and then coupling them together. In the classroom, this provides an excellent educational opportunity for our students since we make analogies between the phase shifts associated with Ampere's Law and equivalent phase shifts in driven oscillators and alternating current RC circuits.

Keywords: electromagnetic waves, conductors, phase delays, physics education

1. Introduction

The study of infinite plane waves propagating through conducting media is standard material in intermediate and advanced textbooks [1–8]. An interesting (but essentially explained) feature is that the magnetic field is delayed in time relative to the electric field. The typical textbook treatment of this issue is primarily mathematical in nature with no physical interpretation of this result. In this work, we will examine how to interpret this phase delay and see how it is equivalent to other physical situations our students have seen.

In section 2, we will review a typical textbook presentation [1]. Those fundamentals certainly lead to many more advanced studies. For example, Boyer contrasts the propagation of electromagnetic waves into a conductor with that of velocity fields [9]. Vitela extends the typical analysis and focuses on the wave polarization and the behavior of the Poynting vector [10] while Shen and Chu provide illustrations exhibiting the range of behavior of reflectance and penetration depth over a very large frequency range [11].

Conversely, this author has been unable to find any educational literature that is directly on point. Li *et al* describe a software package for simulating electromagnetic wave propagation in varying media [12]. This would certainly be a useful visualization tool but it is not focusing on obtaining a better theoretical understanding. There are articles focusing on the understanding of electromagnetic plane waves propagating in a vacuum. For example, Ambrose *et al* studied student misconceptions in that situation [13]. Similarly, Allred *et al* [14] provide a qualitative explanation for why electric fields and magnetic fields in an electromagnetic wave are in phase in a vacuum. They focus on the differential form of Maxwell's equations which is what we will do here as well. However, there do not appear to be any articles in the educational literature related to the propagation of electromagnetic waves in conducting media. This is an open area for investigation.

2. Typical textbook presentation

We have an infinite plane wave propagating in the z-direction. As it crosses the $z = 0$ plane, it passes from a non-conducting medium into a conducting one. We will focus on the wave properties in the conductor and are not interested in what fraction was reflected or transmitted at the boundary. This medium is linear, isotropic and homogeneous and is described by conductivity σ, permittivity ϵ and permeability μ. The free charge density is zero everywhere (so $\rho_f = 0$) and the free current density is described by Ohm's Law (so $\vec{J}_f = \sigma\vec{E}$). Note that the '$f$' for free helps distinguish these charge and current densities from bound ones. We focus on physical situations for which σ, ϵ and μ are all real since that is the case discussed in the texts.

Let us begin by writing Maxwell's equations. After substituting in the linear relationships $\vec{D} = \epsilon\vec{E}$, $\vec{B} = \mu\vec{H}$, $\vec{J}_f = \sigma\vec{E}$ and also $\rho_f = 0$, we get

$$\vec{\nabla} \cdot \vec{E} = 0, \tag{1a}$$

$$\vec{\nabla} \cdot \vec{B} = 0, \tag{1b}$$

$$\vec{\nabla} \times \vec{B} = \mu\sigma\vec{E} + \mu\epsilon\frac{\partial\vec{E}}{\partial t}, \tag{1c}$$

$$\text{and} \quad \vec{\nabla} \times \vec{E} = -\frac{\partial\vec{B}}{\partial t}. \tag{1d}$$

We want to obtain the wave equation so we take the curl of (1d). After using the generic vector identity $\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \nabla^2\vec{E}$ and substituting in (1a) and (1c), we obtain the (scalar) damped wave equation, (2), where $f(z, t)$ represents any component of $\vec{E}$. Similarly, if we take the curl of (1c) and use the same vector identify (for $\vec{B}$) and then substitute in (1*b*) and (1*d*), we obtain (2) for any component of $\vec{B}$. Thus, (2) is the wave equation for all six field components.

$$\frac{\partial^2 f(z,t)}{\partial z^2} - \mu\sigma\frac{\partial f(z,t)}{\partial t} - \mu\epsilon\frac{\partial^2 f(z,t)}{\partial t^2} = 0. \tag{2}$$

If we assume an infinite plane wave solution $f(z, t) = f_0 \exp[\mathrm{i}(kz - \omega t)]$ then (2) yields the dispersion relation

$$k^2 = \omega^2\mu\epsilon + \mathrm{i}\,\omega\mu\sigma. \tag{3}$$

Thus, k is complex and can be written as $k = \alpha + \mathrm{i}\beta$ where β describes the energy absorption by the conductor. If we substitute $k = \alpha + \mathrm{i}\beta$ into (3) and equate real and imaginary parts, we find that

$$\alpha = \omega\sqrt{\frac{\mu\epsilon}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\omega\epsilon}\right)^2}+1\right]^{\frac{1}{2}} \quad \text{and} \quad \beta = \omega\sqrt{\frac{\mu\epsilon}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\omega\epsilon}\right)^2}-1\right]^{\frac{1}{2}}. \tag{4}$$

We now write $\vec{E} = \vec{E}_0 \exp[\mathrm{i}(kz - \omega t)]$ and $\vec{B} = \vec{B}_0 \exp[\mathrm{i}(kz - \omega t)]$ where $\vec{E}_0$ and $\vec{B}_0$ are constant vectors. Since we lost information when we took the curl of (1*c*) and (1*d*), we must substitute these field expressions back into Maxwell's equations (1*a*)–(1*d*) to determine the complete solution. When we do that we obtain

$$E_z = 0, \quad B_z = 0, \quad \hat{z}\times\vec{B} = -\frac{k}{\omega}\vec{E}, \quad \text{and} \quad k\,\hat{z}\times\vec{E} = \omega\vec{B}. \tag{5}$$

These are analogous to the corresponding equations for the non-conducting case so they still tell us that the fields are transverse and that $\vec{E}$, $\vec{B}$ and the propagation direction are mutually perpendicular. However, since k is complex these expressions reveal that $\vec{E}$ and $\vec{B}$ are not in phase.

If we polarize $\vec{E}$ along $\hat{x}$ and let E_0 be a real field amplitude we can write

$$\vec{E} = E_0 \mathrm{e}^{-\beta z}\exp[\mathrm{i}(\alpha z - \omega t)]\,\hat{x} \quad \text{and} \quad \vec{B} = (kE_0/\omega)\mathrm{e}^{-\beta z}\exp[\mathrm{i}(\alpha z - \omega t)]\,\hat{y}. \tag{6}$$

Since we can also write $k = |k|\mathrm{e}^{\mathrm{i}\Omega}$ where

$$\Omega = \tan^{-1}(\beta/\alpha) \tag{7}$$

the *real* fields are

$$\vec{E}_{\text{real}} = E_0\mathrm{e}^{-\beta z}\cos(\alpha z - \omega t)\,\hat{x} \quad \text{and} \quad \vec{B}_{\text{real}} = (|k|E_0/\omega)\mathrm{e}^{-\beta z}\cos(\alpha z - \omega t + \Omega)\hat{y}. \tag{8}$$

Equation (8) illustrates that the magnetic field is delayed in time by the phase angle Ω. In this typical presentation, (4) and (7) describe that phase delay.

The textbooks frequently examine the good conductor/insulator limits. The ratio $\sigma/(\omega\epsilon)$ characterizes how conductive the material is since it is essentially the ratio of the conduction current to the displacement current. For a good insulator ($\sigma \ll \omega\epsilon$), the conduction current is negligible in (1*c*) and $\Omega = 0$. Thus, as expected, the electric and magnetic fields are in phase just like they are in a vacuum. Conversely, for a good conductor ($\sigma \gg \omega\epsilon$) the displacement current is negligible in (1*c*) and $\Omega = \pi/4$ rad.

While many physical situations are described by $\sigma \ll \omega\epsilon$ or $\sigma \gg \omega\epsilon$, there are 'intermediate' conductivity cases where $\sigma/(\omega\epsilon) \simeq 1.0$. For example, microwaves at 2.45 GHz are successfully used in ablation therapy of cancerous tumors (liver, kidney, breast, etc) and they have also been used to detect breast cancer [15, 16]. Additionally, Jiang and Georgakopoulos [17] discuss the possibility of using 3–100 MHz radio signals for communication between an above-ground antenna and robots/water quality monitoring equipment/etc in fresh water. The lower end of that frequency range is in this intermediate conductivity range.

While (4) is useful for comparing the relative size of the real and imaginary parts of k, it obscures the 'source' of the phase delay and provides minimal physical insight. In the rest of this work, we will focus on demystifying this phase delay.

3. A simple mathematical expression for the phase delay

If one is primarily interested in the phase difference, there is a simple expression. Since $k = |k|e^{i\Omega}$, we know that $k^2 = |k^2|e^{2i\Omega}$. Thus, one can use (3) to determine 2Ω. Specifically, $\tan 2\Omega = [\omega\mu\sigma/(\omega^2\mu\epsilon)]$ or

$$\Omega = \frac{1}{2}\tan^{-1}[\sigma/(\omega\epsilon)]. \tag{9}$$

The inherent symmetries of the inverse tangent function reveal some (perhaps unexpected) symmetry. Specifically, the phase angle when $\sigma/(\omega\epsilon) = \frac{1}{100}$ is just as close to zero (0.005 rad) as the phase angle for $\sigma/(\omega\epsilon) = 100$ is to $\pi/4$ rad. Equation (9) provides the (obvious) insight that a non-zero conductivity is needed for there to be a phase delay. The presence of the conduction current helps create this delay. In the next section, we will explore this in significantly more detail.

4. How can we interpret that phase difference?

As we develop a better understanding of the source of this phase difference, we will discover that there are features of this problem that match physical situations our students have encountered in previous courses.

Let us think carefully about how our solutions satisfy Maxwell's equations. Since these transverse infinite plane wave solutions trivially satisfy both versions of Gauss's Law, we can focus on Ampere's Law and Faraday's Law. Specifically, Faraday's Law implies that the curl of $\vec{E}$ is in phase with the time derivative of $\vec{B}$. One might conclude that $\vec{E}$ and $\vec{B}$ must be in phase but that would be incorrect. While both the time and spatial derivatives involve i, the spatial derivative also involves the complex k. Similarly, Ampere's Law implies that the curl of $\vec{B}$ is in phase with the weighted combination of $\vec{E}$ and its time derivative. The overall phase comes from coupling the equations together.

To simplify our analysis, we will consider Ampere's Law and Faraday's Law separately and then combine the results. This is an educational tool that will help us to better understand this phase difference. If an instructor uses this in a classroom, they should make clear to the students that the equations really are coupled together and that one always needs to check to make sure that any solution satisfies all four Maxwell equations. Allred *et al* [14] illustrate the potential risks by highlighting a common textbook problem covering Faraday's Law which is unphysical since Ampere's Law cannot be satisfied. When we solve Ampere's Law and Faraday's Law on their own (in sections 4.1 and 4.2 respectively), we will label the resulting phase difference between the fields (if there is any) as Ω_A (for Ampere's Law) and Ω_F (for Faraday's Law). Once we couple the equations together, the resulting overall phase will be the average of those two phases (see details in section 4.3).

4.1. What if Ampere's Law were a standalone equation?

Let us solve Ampere's Law (1*c*) with the electric field having been *prescribed*. In this situation, we can treat k as a real constant. Let us use the same polarization as in (6) and let $\vec{E} = E_0 \exp[\mathrm{i}(kz - \omega t)]\hat{x}$ and $\vec{B} = \vec{B}_0 \exp[\mathrm{i}(kz - \omega t)]$ where E_0 is real. Substituting into (1*c*), we find that

$$\vec{B}_0 = \frac{(\mathrm{i}\mu\sigma + \omega\mu\epsilon)}{k} E_0\,\hat{y}. \tag{10}$$

On the right-hand side in (10), the conduction current term has an i but the displacement current term does not. Thus, there is a phase difference of $\pi/2$ rad between those two 'sources' leading to a phase difference between the fields. Its magnitude depends upon the relative amplitude of the two terms, $\sigma/(\omega\epsilon)$, which is clearly the same ratio present in (4) and (9). Since the coefficient of E_0 in the displacement current term in (10) is real, it 'wants' the magnetic field in phase with the electric field. Conversely, absorbing the i into the complex exponential, we get $|\vec{B}_{\text{conduction}}| \propto E_0 \exp[\mathrm{i}(kz - \omega t + \pi/2)]$. Thus, the conduction term *alone* would lead to a magnetic field delayed by one-fourth of a period.

Students are frequently surprised by the previous statement. They expect the field to be in phase with the conduction current. It is clear that they have trouble thinking beyond magnetostatics. We remind them that (1*c*) implies that the *curl* of the magnetic field (and not necessarily the magnetic field) is in phase with the current density. In this time-dependent situation, the curl of the magnetic field and the magnetic field are not in phase with each other. Note that Allred *et al* make a related point when discussing electromagnetic waves propagating in a vacuum [14].

So what happens when the conduction current and the displacement current are both included? If we fix z and ignore the common phase from kz, we can rewrite (10) as a real expression for the magnetic field

$$\vec{B}(t) = \left[\left(\frac{\mu\sigma E_0}{k}\right)\sin\omega t + \left(\frac{\mu\omega\epsilon E_0}{k}\right)\cos\omega t\right]\hat{y}. \tag{11}$$

We then combine the sine and cosine in (11) to get

$$\vec{B}(t) = \frac{\mu E_0\sqrt{\sigma^2 + \omega^2\epsilon^2}}{k}\cos(\omega t - \Omega_A)\hat{y} \quad \text{where} \quad \Omega_A = \tan^{-1}[\sigma/(\omega\epsilon)]. \tag{12}$$

Thus, Ampere's Law would lead to the magnetic field delayed (relative to the electric field) by a phase Ω_A. Equivalently, this introduces the complex factor $\exp(\mathrm{i}\Omega_A)$ into k^2.

Since the magnetic field was determined by two sources one-fourth of a cycle apart, this problem is mathematically/conceptually equivalent to many other physics problems our students have seen. Ask your students if they can recall any matching physical situations. We will briefly examine two here. Discussing either one in the classroom would help our students transfer their prior knowledge and help them better understand this complex problem.

4.1.1. A driven oscillator. Let us drive a mass with two oscillating forces: one described by a cosine and one described by a (negative) sine. The two forces are analogous to the conduction and displacement currents in (1*c*). Newton's 2nd Law and its solution are

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = F_0\cos\omega t - F_1\sin\omega t \quad \rightarrow\rightarrow\rightarrow \quad v(t) = \frac{F_0}{m\omega}\sin\omega t + \frac{F_1}{m\omega}\cos\omega t. \tag{13}$$

The integration constant in $v(t)$ has been set to zero and the solution is analogous to (11). The velocity corresponds to the magnetic field while $F_0 \propto \sigma$ and $F_1 \propto \omega\epsilon$ (ignoring the common factor $\mu E_0/k$ in (11) and $m\omega$ in (13)). To complete the analogy for comparison with (12), we rewrite this to illustrate the velocity's phase delay (relative to the expected oscillation phase due to F_1):

$$v(t) = \frac{\sqrt{F_0^2 + F_1^2}}{m\omega}\cos(\omega t - \phi) \quad \text{where} \quad \phi = \tan^{-1}(F_0/F_1). \tag{14}$$

4.1.2. A series alternating current RC circuit. We could also analyze a series alternating current circuit with a resistor and a capacitor. If we write $I(t) = I_{\max}\cos\omega t$, then Kirchhoff's loop rule tells us that the source voltage $V(t)$ must be

$$V(t) = \Delta V_C(t) + \Delta V_R(t) = \frac{1}{C}\int I(t)\mathrm{d}t + I(t)R = \frac{1}{\omega C}I_{\max}\sin\omega t + I_{\max}R\cos\omega t. \tag{15}$$

In (15), $\Delta V_C(t)$ and $\Delta V_R(t)$ are positive when there is a voltage drop across them and the integration constant has been set to zero. Equation (15) is also analogous to (11) where the source voltage corresponds to the magnetic field and $1/(\omega C) \propto \sigma$ and $R \propto \omega\epsilon$. Calculating the phase delay between the AC source voltage and the current is identical to calculating the magnetic field phase delay relative to the electric field in the conductor. To complete the analogy, we rewrite (15) to illustrate the phase delay of the source voltage relative to the current:

$$V(t) = I_{\max}\sqrt{\frac{1}{\omega^2C^2} + R^2}\,\cos(\omega t - \delta) \quad \text{where} \quad \delta = \tan^{-1}[1/(\omega RC)]. \tag{16}$$

4.2. What if Faraday's Law were a standalone equation?

Notice that Ω_A (12) is exactly twice Ω in (9). This reminds us that Ampere's Law is not a standalone equation. It is coupled to Faraday's Law (1*d*). Thus, let us now solve Faraday's Law with the electric field having been *prescribed* where again we can treat k as a real constant. Substituting the same expressions for $\vec{E}$ and $\vec{B}$ into (1*d*) we find that

$$\vec{B}_0 = (kE_0/\omega)\hat{y}. \tag{17}$$

Since the coefficient of E_0 is real, this is just like the effect of the displacement current in (10) so we expect the magnetic field to be in phase with the electric field (just as they are when $\sigma = 0$.) Effectively, Faraday's Law introduces a factor $\exp(\mathrm{i}\Omega_\mathrm{F})$ into k^2 where $\Omega_\mathrm{F} = 0$.

4.3. Now let us couple Faraday's Law and Ampere's Law

We now combine the results of 4.1 and 4.2. When Faraday's Law and Ampere's Law are coupled together in (2), the phase of k^2 is determined by the product of $\exp(\mathrm{i}\Omega_\mathrm{F})$ and $\exp(\mathrm{i}\Omega_A)$. Thus, the phase of k is determined by the average of the phases, $\Omega = (\Omega_F + \Omega_A)/2$, or equivalently

$$\Omega = \frac{1}{2}\tan^{-1}[\sigma/(\omega\epsilon)]. \qquad \text{(9 revisited)}$$

The analyses in 4.1 and 4.2 treated k as if it was real but it is complex in the final solution. Once its true nature is included, both Faraday's Law and Ampere's Law clearly predict that *same* phase difference between the fields.

Recognizing that $\Omega = (\Omega_{\mathrm{F}} + \Omega_A)/2$ also helps us interpret the good conductor limit discussed earlier. When $\sigma \gg \omega\epsilon$, $\Omega_{\mathrm{F}} = 0$ and $\Omega_A = \pi/2$ rad so $\Omega = \pi/4$ rad. Faraday's Law alone would result in the fields being in phase while Ampere's Law would require a $\pi/2$ rad delay so after coupling them that delay is $\pi/4$ rad. This is consistent with the electric field being associated with the changing magnetic field and the magnetic field being associated with the conduction current.

Additionally, nothing prevents us from including an imaginary component in σ and/or ϵ and/or μ. This changes the value of Ω_A but the overall phase delay is still $\Omega = (\Omega_{\mathrm{F}} + \Omega_A)/2$. For example, if we let $\sigma = \sigma_R + \mathrm{i}\sigma_I$ and leave ϵ and μ real, we get $\Omega = \frac{1}{2}\tan^{-1}[\sigma_R\big/(\omega\epsilon - \sigma_I)]$.

5. Conclusion

In this work, we have analysed the phase delay of the magnetic field relative to the electric field for an infinite plane wave propagating through a conducting medium. We treat Faraday's Law and Ampere's Law as if they were standalone equations and then couple their effects together. This method of treating coupled equations could be beneficial for students in many situations. We show that calculating the magnetic field using Ampere's Law is analogous to calculating the velocity of a mass driven by two oscillating forces which are one-fourth of a cycle apart and also to calculating the AC source voltage in a series RC circuit. Once we couple it with Faraday's Law, we obtain the correct phase relationship between the fields in a very convenient form. This helps demystify the textbook expressions for the phase angle. Since the textbooks often provide explanations for related features of this problem (e.g. why the conduction current makes the magnetic field energy larger than the electric field energy), this analysis supplements that and helps improve students' physical understanding of this problem.

Acknowledgments

We thank the anonymous referees for their useful suggestions and for catching an error in the references.

Data availability statement

There isn't any data for this paper.

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