The Accuracy of Series and Parallel Connections of Four-Terminal Resistors

JACK C. RILEY, SENIOR MEMBER, IEEE

Abstract—The range and accuracy of resistance calibration can be increased by the use of series and parallel connections of four-terminal resistors. Low value resistors can be permanently connected in series and reconnected in parallel by using Hamon's[1] technique to change resistance level without materially affecting resistance accuracy. The resistors are connected in parallel by attaching shorting bars to one terminal at each end of each resistor and attaching matched resistors in series with the other terminals. High accuracy can be attained even though lead and connection resistance are relatively high.

The purpose of this paper is to provide a theoretical base and an error analysis to justify the use of the series-to-parallel transfer technique at low resistance levels. The analysis uses a four-terminal equivalent circuit suggested by Searle.[2] The accuracy of series and parallel connections of groups of like resistors is investigated in terms of the equivalent circuit. Procedures are developed for determining the connection accuracy of a set of resistors in parallel or series.

Resistance Transfer

A CONVENIENT technique for confidently transferring from one accurately known resistance value to other levels is the use of equal value resistors in series and parallel configurations. For example, ten equal resistors in series have one hundred times the resistance of the same resistors in parallel. It is easy to see how an accurate change in level can be made with perfect resistors having zero-resistance connections and no leakage resistance paths. Unfortunately these conditions can only be approached, not met. The real advantage of the series-to-parallel connection change is found when using less than perfect resistors. In fact, the series-to-parallel technique is so accurate that even for the most precise transfers the two configurations can usually be assumed to have the same accuracy. This technique will also work for very low value resistors if special junction configurations and four-terminal connections are used. High value resistance transfers can be accomplished accurately by using three-terminal techniques and low leakage designs. Thus, a tremendous range of resistance values can be accurately intercompared.

So far, this discussion has been qualitative. A quantitative analysis of the accuracy limitations shows how far the assumption of equal series and parallel accuracy can be trusted. The analysis is broken into several phases.

First, the accuracy of the series-to-parallel change is investigated assuming perfect connections between resistors which are not quite equal in value.

Next, junction resistance is investigated and a true four-terminal junction is proposed. The effect of junction inaccuracies is then investigated for both series and parallel connections. The results of these investigations are reduced to test measurements which can be used to predict the expected accuracy.

Manuscript received May 4, 1965; revised March 10, 1967.

The author is a Technical Consultant to Electro Scientific Industries, Portland, Ore.

[1] B. V. Hamon, *J. Sci. Instr.*, vol. 31, no. 12, December 1954.

[2] G. F. C. Searle, *Electrician* (London), vol. 66, 1911, p. 999.

Equal Resistors Connected in Series and Parallel

Our object is an investigation of the effect of combining nominally equal resistors which differ from each other slightly. The nominal value of each resistor is R, but each will deviate from R by Δ_n proportional parts.

$$R_n = R(1 + \Delta_n), \tag{1}$$

where R_n is the value of the nth resistor in ohms, R is the nominal value of each resistor in ohms, and Δ_n is the deviation of R_n from R in proportional parts.

For this part of the discussion we shall assume that perfect junctions are possible and that the values of individual resistors are the same whether connected in series or parallel.

Series Value: If m of these nominally equal resistors are connected in series (Fig. 1), the total resistance will

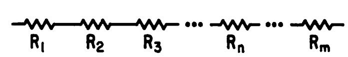


Fig. 1. Resistors in series.

be nominally equal to mR but it will be slightly different by an amount Δ_{av}. This can be shown algebraically

$$\begin{aligned} R_S &= \sum_{n=1}^{m} R_n \\ &= \sum_{n=1}^{m} R(1 + \Delta_n) \\ &= mR + R\sum_{n=1}^{m} \Delta_n \\ &= mR\left(1 + \frac{1}{m}\sum_{n=1}^{m} \Delta_n\right), \end{aligned} \tag{2}$$

where R_S is the total resistance of m resistors R_n in series, in ohms,

$$\Delta_{av} \equiv \frac{1}{m}\sum_{n=1}^{m} \Delta_n, \tag{3}$$

where Δ_{av} is the average of the deviations of R_n from R, in proportional parts. Thus, the series resistance can be expressed in terms of its expected value and the average deviation

$$R_S = mR(1 + \Delta_{av}). \tag{4}$$

Parallel Value: If these same m resistors are reconnected in parallel (Fig. 2), the circuit equation will be

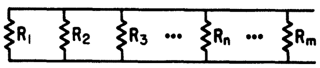


Fig. 2. Resistors in parallel.

more complicated, but it can be reduced to a similar expression

$$R_p = \frac{1}{\sum_{n=1}^{m} \frac{1}{R_n}} = \frac{1}{\sum_{n=1}^{m} \frac{1}{R(1+\Delta_n)}}, \tag{5}$$

where R_p is the total parallel resistance of m resistors R_n in ohms. In this form it is hard to see any usable relation between the series and parallel equations. Two series expansions will give the usable form and indicate the difference between the series and parallel accuracy. Since

$$\frac{1}{1+\Delta} = 1 - \Delta + \Delta^2 - \Delta^3 + \cdots \tag{6}$$

then

$$\sum_{n=1}^{m} \frac{1}{R(1+\Delta_n)} = \frac{1}{R} \sum_{n=1}^{m} (1 - \Delta_n + \Delta_n^2 - \Delta_n^3 + \cdots) = \frac{m}{R}\left[1 - \frac{1}{m}\sum_{n=1}^{m} (\Delta_n - \Delta_n^2 + \Delta_n^3 - \cdots)\right], \tag{7}$$

and

$$\begin{aligned} R_p &= \frac{1}{\frac{m}{R}\left[1 - \frac{1}{m}\sum_{n=1}^{m} (\Delta_n - \Delta_n^2 + \Delta_n^3 - \cdots)\right]} \\ &= \frac{R}{m}\left\{1 + \frac{1}{m}\sum_{n=1}^{m} (\Delta_n - \Delta_n^2 + \Delta_n^3 - \cdots)\right. \\ &\quad + \left[\frac{1}{m}\sum_{n=1}^{m} (\Delta_n - \Delta_n^2 + \Delta_n^3 - \cdots)\right]^2 \\ &\quad \left. + \left[\frac{1}{m}\sum_{n=1}^{m} (\Delta_n - \Delta_n^2 + \Delta_n^3 - \cdots)\right]^3 + \cdots\right\}. \end{aligned} \tag{8}$$

The Δ, Δ^2, and Δ^3 terms can be collected to find the contribution of each order

$$\begin{aligned} R_p &= \frac{R}{m}\left\{1 + \frac{1}{m}\sum_{n=1}^{m} \Delta_n \right. \\ &\quad - \left[\frac{1}{m}\sum_{n=1}^{m} \Delta_n^2 - \left(\frac{1}{m}\sum_{n=1}^{m} \Delta_n\right)^2\right] \\ &\quad + \left[\frac{1}{m}\sum_{n=1}^{m} \Delta_n^3 - 2\left(\frac{1}{m}\sum_{n=1}^{m} \Delta_n\right)\left(\frac{1}{m}\sum_{n=1}^{m} \Delta_n^2\right) \right. \\ &\quad \left.\left. + \left(\frac{1}{m}\sum_{n=1}^{m} \Delta_n\right)^3\right] - \cdots\right\} \\ &= \frac{R}{m}\left\{1 + \Delta_{av} - \left[\frac{1}{m}\sum_{n=1}^{m} \Delta_n^2 - \Delta_{av}^2\right]\right. \\ &\quad + \left[\frac{1}{m}\sum_{n=1}^{m} \Delta_n^3 - 2\Delta_{av} \right. \\ &\quad \left.\left. \cdot \left(\frac{1}{m}\sum_{n=1}^{m} \Delta_n^2\right) + \Delta_{av}^3\right] - \cdots\right\}. \end{aligned} \tag{9}$$

If second- and higher-order terms are eliminated, the most often used expression for the parallel connection results:

$$R_p \approx \frac{R}{m}(1 + \Delta_{av}). \tag{10}$$

The second-order, Δ^2, terms exceed all of the higher-order terms so they give a measure of the accuracy of the simple expression for R_p given by (10). Manipulation of the second-order terms brings out an interesting relation between them and the sample variance of the resistance values involved:

$$\begin{aligned} \frac{1}{m}\sum_{n=1}^{m} \Delta_n^2 - \Delta_{av}^2 &= \frac{1}{m}\sum_{n=1}^{m} \Delta_n^2 - 2\Delta_{av}\left(\frac{1}{m}\sum_{n=1}^{m} \Delta_n\right) + \Delta_{av}^2 \\ &= \frac{1}{m}\sum_{n=1}^{m} [\Delta_n^2 - 2\Delta_{av}\Delta_n + \Delta_{av}^2] \\ &= \frac{1}{m}\sum_{n=1}^{m} (\Delta_n - \Delta_{av})^2. \end{aligned} \tag{11}$$

The second-order terms combine to give the expression for the sample variance, or square of the sample standard deviation, of the values of Δ. The parallel resistance will be lower than predicted by (10) by the variance of the deviation of the resistors from their average value. Third- and higher-order terms will have an exceedingly small influence on the value.

A simple example will show what effect these high-order terms might cause. If ten resistors, five of which are 0.1 percent high and five of which are 0.1 percent low, are connected in parallel, the expected value can be calculated. The conductance values will be 0.1001001 $\cdots$ percent high and 0.099900099 $\cdots$ percent low. These will combine to give a resistance that is low by one ppm to the first eleven places. Equation (10) predicts no error. Equation (11) predicts the one-ppm low, the third-order expression in (9) predicts zero, and the fourth-order term would add two parts in 10^{12}, etc.

$$\frac{1}{m}\sum_{n=1}^{m}(\Delta_{av}-\Delta_n)^2 = \frac{1}{10}\sum_{n=1}^{m}(0-\Delta_n)^2 = \frac{1}{10}\sum_{n=1}^{m}[(10)(10^{-3})^2] = 10^{-6} = 1 \text{ ppm} \tag{12}$$

$$\frac{1}{10}\sum_{n=1}^{m}\Delta_n^3 = \frac{1}{10}[(5)(10^{-3})^3-(5)(10^{-3})^3] = 0 \tag{13}$$

$$\frac{1}{10}\sum_{n=1}^{m}\Delta_n^4+\left[\frac{1}{10}\sum_{n=1}^{m}\Delta_n^2\right]^2 = \frac{1}{10}[10(10^{-3})^4]+10^{-12} = 2 \text{ in } 10^{12}. \tag{14}$$

For part-per-million accuracy of the series–parallel transfer, the resistors must be alike within a sample deviation of less than 0.1 percent. If this is the case, (10) can be used safely, and the series and parallel values will differ less than one ppm.

Four-Terminal Resistors

Greater resistance consistency and accuracy can be realized if lead and contact resistances can be removed from the measurement circuit. This can be done by making separate voltage and current leads on each end of a resistor. By measuring the voltage potentiometrically so that no current flows through the potential leads, the direct resistance value can be properly determined. Four-terminal resistors have a fixed value of direct resistance even when current is being drawn through the potential leads. Usually, the potential and current leads can be interchanged in several ways without changing the value of the direct four-terminal resistance. These last two statements are easy to believe when the resistor has long leads coming from small junctions, but in the general case of four leads randomly connected to a piece of conductor, some questions begin to arise. To make four-terminal junctions these questions have to be answered. A simple equivalent circuit for any four-terminal configuration can be derived by using the superposition and reciprocity theorems. The measurement characteristics of the equivalent circuit are just as valid and much more obvious than those of the true configuration.

Generalized Resistor: The resistor that is to be analyzed can be a network or a conductive blob. Four terminals are connected to it. It can be heterogeneous and nonisotopic, but it cannot exhibit any rectifying action, and the resistivity of all its parts must be constant at all voltages and currents to which they will be subjected. At a constant temperature most metals meet these requirements but many semiconductors do not. Such a resistor is shown in Fig. 3.

Equivalent Circuit: The performance of this generalized four-terminal resistor is not very obvious. A more easily analyzed simple equivalent circuit, which has exactly the same characteristics, can be developed by investigating all of the possible voltage and current relations that can be measured at the terminals. This can be done by supplying current to one terminal and taking it out of another in each of the twelve possible ways. If an equivalent circuit can be devised which will give the same voltages at all four terminals for each current condition, it will be an exact equivalent. This is true because of the superposition theorem.

Superposition Theorem: The current that flows at any point in a linear network, or the potential difference which exists between any two points in such a network, due to the simultaneous action of a number of EMF's distributed in any manner throughout the network, is the sum of the component currents at the first point, or the potential difference between the two points that would be caused by the individual EMF's acting alone.

Measurable Resistances: For each current condition there are twelve measurable voltages, as shown in Fig. 4. Each of the six voltages shown represent two because they can be measured in either direction. Twelve currents with twelve voltages each, result in 144 distinct resistances which must be duplicated. This number can be reduced sharply by noting that reversing the current will reverse the voltage direction without changing any resistance, and that changing the direction of measuring the voltage will change its sign in such a way that again the resistance value will be preserved. These considerations leave us with the thirty-six measurable resistances shown in the six equivalent circuits of Fig. 5. The four subscripts on each resistor are: first, the terminal that the current enters; second, the voltage reference terminal; third, the voltage measurement terminal; and fourth, the terminal that the current leaves. Many of these thirty-six resistors are alike.

Reciprocity Theorem: If an EMF of any character whatsoever located at one point in a linear network produces a current at any other point in the network, the same EMF acting at the second point will produce the same current at the first point. The reciprocity theorem allows us to interchange the voltage and current terminals. Resistors with the inner and outer subscripts interchanged will be the same. Also complete reversal of the subscripts indicates that both the current and voltage are changed and, therefore, there is no change in the resistance value. For example,

$$R_{ABCD} = R_{DCBA} = R_{BADC} = R_{CDAB}. \tag{15}$$

These considerations leave the twenty-one unique resistances shown in Fig. 6. Each of the missing ones is just like one of these twenty-one. Each of the six equivalent circuits has only three unique resistances. The other

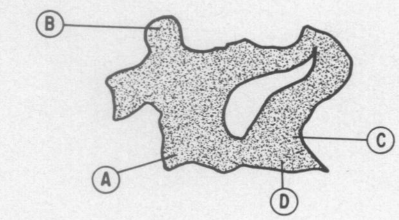


Fig. 3. Generalized four-terminal resistors.

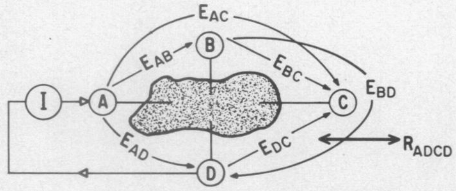


Fig. 4. Voltages for each current.

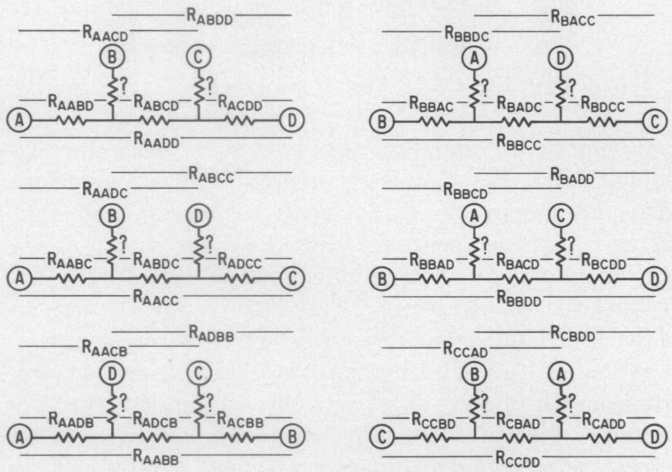


Fig. 5. Six equivalent circuits.

R_{AABD}	R_{ABCD}	R_{ACDD}	R_{AACD}	R_{ABDD}	R_{AADD}
R_{AABC}	R_{ABDC}	R_{ADCC}	R_{AACC}	R_{ABCC}	
R_{ADCB}	R_{ACBB}	R_{ADBB}	R_{AABB}		
R_{BDCC}	R_{BBDC}	R_{BBCC}			
R_{BCDD}	R_{BBDD}				
R_{CCDD}					

Fig. 6. Unique resistances.

$R_{AABD} = R_A$	$R_{CCBD} = R_C$	$R_{ABCD} = C$
$R_{BBAC} = R_B$	$R_{ACDD} = R_D$	$R_{ABDC} = D$

Fig. 7. Six independent resistances.

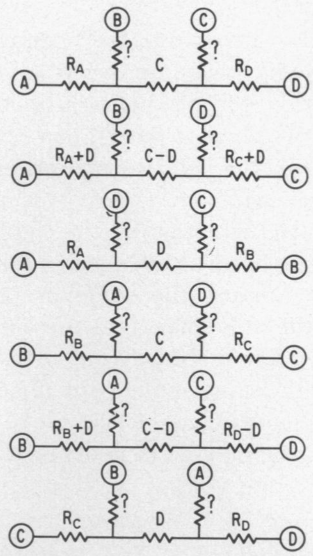


Fig. 8. Six resistances make all six equivalent circuits.

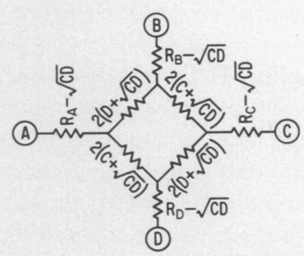


Fig. 9. Four-terminal resistor equivalent circuit.

three are made of combinations of those three. For example,

$$R_{AACD} = R_{AABD} + R_{ABCD}. \tag{16}$$

Making these substitutions leaves us with only six independent resistances. This implies that a six resistor equivalent circuit might be possible. The simplest one which will work is a tetrahedron with a resistor along each edge. The circuit suggested by Searle[2] will be used instead because it is more convenient to use in later calculations. To work toward the desired circuit six resistance values will be chosen, as shown in Fig. 7. These six resistances can be used to make all of the resistances in the six equivalent circuits of Fig. 5. This is shown in Fig. 8.

A Single Equivalent Circuit: The circuit of Searle is shown in Fig. 9. This circuit with the values indicated will give all of the values shown in Fig. 8. The values of Fig. 8 will give the values in Fig. 5 which can be used in turn to represent all of the possible measurements on a four-terminal circuit. Thus, the circuit of Searle will represent the four-terminal unit exactly. In addition, three circuits can be chosen from Fig. 8 which permit direct measurements of the six resistance values shown in Fig. 7. The first, third, and fourth circuits of Fig. 8 are suggested.

A Four-Terminal Standard Resistor: The four-terminal standard resistor is a special case of the generalized unit of Figs. 3 and 9. The direct resistance D is the one of interest, and the cross resistance C is so small that it can be neglected. As C approaches zero the equivalent circuit approaches that of Fig. 10. With this circuit it makes no difference whether the voltage and current leads are reversed at one or both ends. Searle[2] investigated the design of current shunts to find when the equivalent circuit of Fig. 10 could be used. To make the

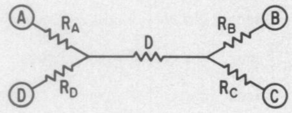


Fig. 10. Four-terminal standard resistor.

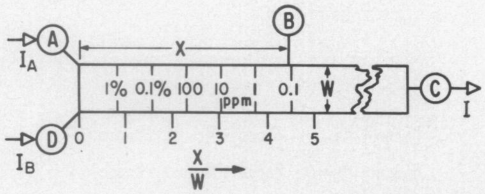


Fig. 11. Resistance change caused by changing voltage and current terminals at one end.

analysis he chose a "worst case" configuration which could be mathematically analyzed and also provide useful information for predicting the behavior of an actual resistor. He chose a flat ribbon of resistance material, as shown in Fig. 11. Normal operation would be with current flowing from D to C, using A and B as potential leads. He investigated the change in potential at B when the current was changed to flow from A to C. If the current remains the same in both cases, this change at B is the only change in the voltage across the direct resistance. The voltage from A to C with current entering at D is the same as the voltage between D and C with the current entering at A because of the reciprocity theorem. Thus, the change in resistance is proportional to the change in the voltage at B when the current is moved from A to D. For one ppm the resistor need only be four times as long as it is wide. The values shown are voltage change in proportional parts of the voltage drop along a length of the ribbon equal to its width and carrying the same current as that entering A or D. For a cylinder, such as a piece of wire, the resistance change will be even smaller. If there is any question about the change it can be easily measured. From the circuit of Fig. 9, the change can be found by measuring the cross resistance C. The cross resistance is the ratio of the voltage between terminals B and C to the current entering A and leaving D. This measured cross resistance will be the amount of the difference between the four-terminal resistances measured with the current and potential leads at one end, first normal and then interchanged. No matter what the resistor configuration, C will be the same whether the interchange is made on one end or the other. These results can be verified by examining Fig. 8.

A Four-Terminal Junction: It would often be desirable to have four wires emerging from exactly the same point electrically. Fortunately, this can be done to any desired degree of precision with finite and rather simple junctions.

A four-terminal junction results when both the cross and direct resistances of the equivalent circuit are zero.

If current can be passed between two of the four terminals without producing a voltage between the other two, in all possible ways, the unit is a four-terminal junction. Actually, any two such measurements which show that C and D are both zero are sufficient. The equivalent circuit for such a junction and some physical arrangements for realizing it are shown in Fig. 12. A practical four-terminal junction used in the ESI Model SR 1010 Resistance Standard is shown in Fig. 13. The direct and cross resistances of these junctions are less than 0.1 $\mu\Omega$. This is the resistance of about one one-thousandth of an inch of number ten copper wire. The notch in the end of the junction can be filed to make the cross and direct resistances even smaller. Hamon[1] describes another practical configuration consisting of a cylinder with one coaxial lead and three equally spaced radial leads.

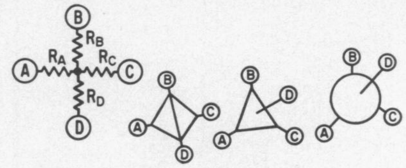


Fig. 12. Four-terminal junctions.

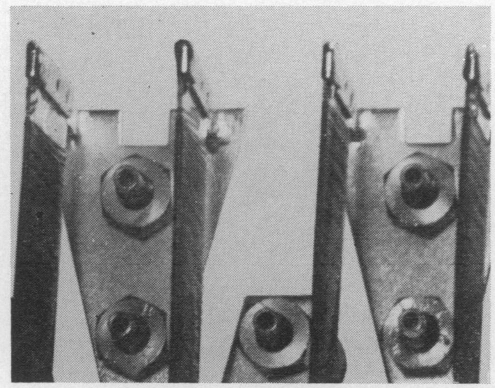


Fig. 13. A four-terminal junction.

Four-Terminal Resistors in Series

If a group of resistors are connected in series by four-terminal junctions, any series combination can be measured by four-terminal techniques. If the junctions are truly zero-resistance four-terminal junctions, the series resistances will be sums of the individual resistances. If this is true the group can be used as a resistance transfer device. Individual resistances can be measured and their sums calculated. The sums can then be used as known standards at other resistance levels.

The accuracy of two resistors in series is investigated by means of the circuit of Fig. 14. Here, two resistors are connected by a four-terminal junction which is indicated by its equivalent circuit. This circuit can be analyzed to see the difference between four-terminal measurements of the series resistance and the sum of the two individual resistances. In the series measurement the center resistances combine in parallel to give the circuit shown in Fig. 15. For the individual measurements, the center resistances form delta circuits which can be replaced by equivalent wye circuits to give the results

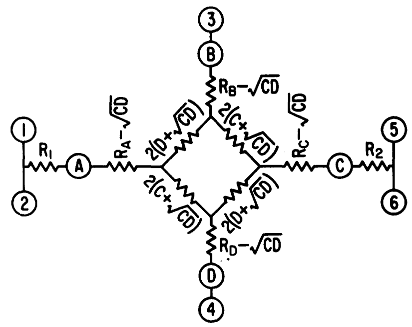


Fig. 14. Two resistors connected in series by a four-terminal junction.

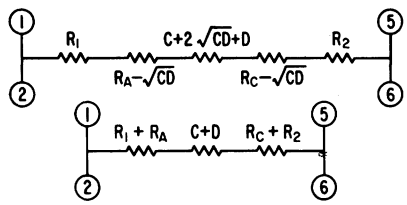


Fig. 15. Series measurement.

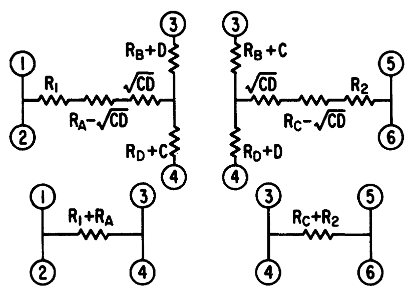


Fig. 16. Individual measurements.

shown in Fig. 16. The only difference between the series measurement and the sum of the individual measurements is the sum of the cross and direct resistance values of the four-terminal junction. These values can be measured to find whether or not they can be ignored. If they are large enough to be a problem the measured values can be used for making a correction. The measurement circuits for C and D are shown in Fig. 17. The measurements can be made by the volt/ampere method or directly with a Kelvin bridge.

There are other connections for making this same measurement. The plus battery lead can be connected to the same terminal for both measurements. The minus voltmeter lead does not need to be moved either. If the values of C and D are going to be used for corrections, the polarity of the voltage and current must be observed. The polarity shown is for a positive value of resistance, but the four-terminal resistance measured is quite likely to be negative. Of course, reversing both current and voltage directions results in the same resistance polarity. If the measurement is being made with a Kelvin bridge it will be impossible to balance the negative resistance. If trouble is encountered, reverse the potential leads and try again.

If a group of resistors is connected in series by four-

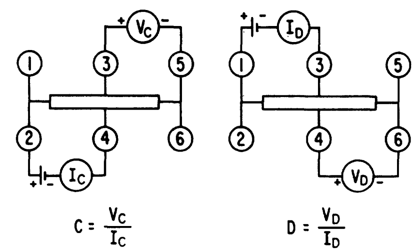


Fig. 17. Measuring C and D.

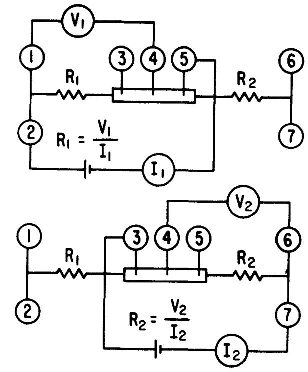


Fig. 18. Resistors connected in series by five-terminal junctions.

terminal junctions, each junction will contribute its values of cross and direct resistance to the series measurement but not to the individual measurements. If each cross and direct junction resistance is less than some limiting value M, the difference between the sum of the individual measurements and the series measurement can be expressed by

$$mR\left[1 + \Delta_{av} - 2\frac{M(m-1)}{Rm}\right] \le R_S \le mR = \left[1 + \Delta_{av} + 2\frac{M(m-1)}{Rm}\right]. \qquad (17)$$

If ten 10-ohm resistors are connected with junctions like those of Fig. 13 for which M is $\frac{1}{2}$ $\mu\Omega$, the resulting series resistance will be given by

$$(10)(10)\left[1 + \Delta_{av} \pm 2\frac{5 \times 10^{-7}(10-1)}{10(10)}\right] = 100[1 + \Delta_{av} \pm 0.09 \text{ ppm}], \qquad (18)$$

where Δ_{av} is found from individual measurements of the resistors. If greater accuracy is needed the individual values of C and D for the junctions can be measured and used to find a much closer value of the series resistance.

Five-Terminal Junctions: If a series of four-terminal resistors does not have to be connected in parallel a much simpler and more reliable junction can be made. By making three connections to a connecting conductor that is several times as long as it is wide the center junction point will not move a measurable amount when the individual resistances are measured, as shown in Fig. 18. The series combination will be given very accurately by the sum of these measurements.

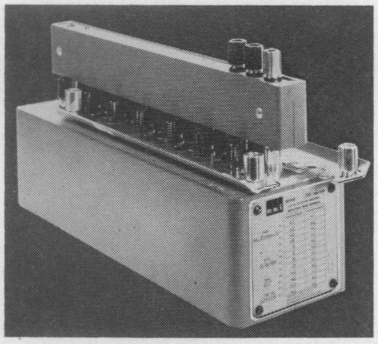


Fig. 19. Resistance standard with shorting bars and network.

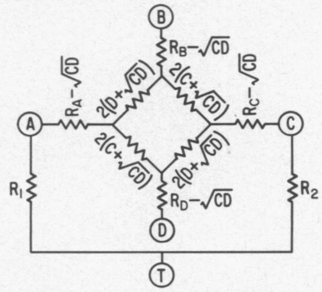


Fig. 20. Two parallel connected resistors joined by a four-terminal junction.

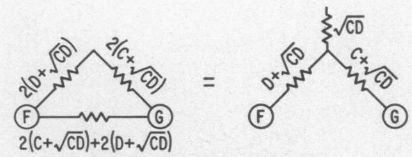


Fig. 21. Delta-wye transformation of junction center.

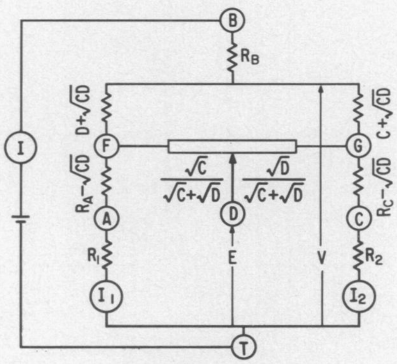


Fig. 22. Equivalent parallel circuit

Four-Terminal Resistors in Parallel

Four-terminal junctions and special four-terminal connections can be combined to make parallel connections to four-terminal resistors in a way that produces almost no connection error. Hamon[1] has described such a resistance device. The ESI model SR 1010 resistance box connected by the PC 101 parallel compensation network and SB 103 shorting bars shown in Fig. 19 is another. Confidence in the use of this simple system can be gained by analyzing the source of possible errors and determining their expected magnitude.

Parallel Connection with Four-Terminal Junctions: If two resistors are connected by a four-terminal junction and if it is assumed that the other ends of the resistors are connected to perfect junctions, the circuit of Fig. 20 will result. This circuit can be analyzed to find the difference between the measured parallel resistance and the desired parallel combination of the two measured resistances. Assuming no error at the other end, the first step in the analysis is to change the delta circuit of the center junction resistors into an equivalent wye, as shown in Fig. 21. The resulting circuit is shown in Fig. 22. The parallel resistance will be determined by assuming that a current enters at B and leaves at T. The measured resistance is the voltage E from D to T divided by this current. The voltage at D will be the same as though an infinite impedance divider had been connected between F and G and set as shown. This can be seen from Fig. 20. The voltage from the equivalent wye junction to T is V. The currents I_1 through R_1 and I_2 through R_2 can be found in terms of V.

$$I_1 = \frac{V}{R_1 + R_A + D}$$

$$I_2 = \frac{V}{R_2 + R_C + C}, \tag{19}$$

and combined to give the total current I.

$$I = V\left(\frac{1}{R_1 + R_A + D} + \frac{1}{R_2 + R_C + C}\right).$$

Thus, the voltage of F and G can be found, and finally E can be determined,

$$E = \frac{I_1\sqrt{D}(R_1 + R_A - \sqrt{CD}) + I_2\sqrt{C}(R_2 + R_C - \sqrt{CD})}{\sqrt{C} + \sqrt{D}}, \tag{20}$$

to give the total parallel measured resistance

$$R_P = \frac{\sqrt{D}\,\dfrac{R_1 + R_A - \sqrt{CD}}{R_1 + R_A + D} + \sqrt{C}\,\dfrac{R_2 + R_C - \sqrt{CD}}{R_2 + R_C + C}}{(\sqrt{C} + \sqrt{D})\left(\dfrac{1}{R_1 + R_A + D} + \dfrac{1}{R_2 + R_C + C}\right)}$$

$$= \frac{(R_1 + R_A)(R_2 + R_C)}{R_1 + R_A + R_2 + R_C} \cdot \frac{1 - \dfrac{CD}{(R_1 + R_A)(R_2 + R_C)}}{1 + \dfrac{C + D}{R_1 + R_A + R_2 + R_C}}. \tag{21}$$

The individual resistance measurements are the same that they were for the series case. They combine in parallel to give the desired resistance

$$R_Q = \frac{(R_1 + R_A)(R_2 + R_C)}{R_1 + R_2 + R_A + R_C}\cdot \tag{22}$$

The difference between (21) and (22) is the error caused by an imperfect junction.

$$R_P - R_Q = -\frac{R_Q(C + D) + 2CD}{R_1 + R_2 + R_A + R_C + C + D}\cdot \tag{23}$$

Resistances C and D are usually quite small so their product can be ignored. If the individual measured values are almost alike, (23) reduces to

$$R_P - R_Q \approx -\frac{C + D}{2} \tag{24}$$

for practical values. The resulting deviation of the parallel measurement from that predicted by the individual measurements in terms of the value of the individual resistors, R, being connected in parallel is

$$R_P \approx R_Q\left(1 - \frac{C + D}{2R}\right). \tag{25}$$

If neither C nor D exceeds M in value, (10) and (25) can be combined to give

$$R_P \approx \frac{R}{2}\left(1 + \Delta_{\text{av}} \pm \frac{M}{R}\right), \tag{26}$$

where R_p is the measured parallel resistance, in ohms, R is the nominal value of resistors connected in parallel, in ohms, Δ_{av} is the average deviation of individual measured resistance values from nominal value R, in proportional parts, and M is the largest junction error, in ohms. Thus, the error resulting from the parallel connection of two resistors, which are already connected in series by a four-terminal junction, will be the ratio of the sum of the direct and cross resistances of the junction to twice the value of the individual resistors. For two 10-ohm resistors with M equal to $\frac{1}{2}$ $\mu\Omega$, the parallel connection error would be five parts in ten to the eighth. Thus far, it has been assumed that the opposite ends of the series pair were connected perfectly. This connection is investigated next.

Parallel Connection With Shorting Bar and Network: The second problem in combining a group of four-terminal resistors in parallel is the connection when the resistors are not already joined by a four-terminal junction. Somehow leads must be taken from all of the resistors to the current and potential terminals of the measuring circuit. The resulting circuit for joining two resistors, assuming that they already have a perfect junction on one end, is shown in Fig. 23. From this circuit the equation for the measured four-terminal resistance of the parallel combination can be derived in

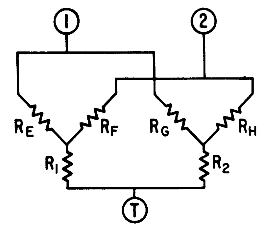
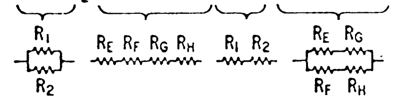


Fig. 23. Two resistors in parallel.

much the same manner that was used for the paralleling junction. The resulting parallel resistance is given in

$$R_P = \left(\frac{R_1R_2}{R_1+R_2}\right)\left[1 + \overbrace{\frac{(R_1 R_2)\left(\frac{R_H}{R_2} - \frac{R_F}{R_1}\right)\left(\frac{R_G}{R_2} - \frac{R_E}{R_1}\right)}{(R_E+R_F+R_G+R_H)\left(R_1+R_2 + \frac{(R_E+R_G)(R_F+R_H)}{(R_E+R_F+R_G+R_H)}\right)}}^{\text{ERROR TERM}}\right] \tag{27}$$



For convenience in understanding their influence, equivalent circuits have been drawn for appropriate parts of the equation. The parallel combination of R_1 and R_2 is the desired result. The right hand part of the equation should be as near zero as possible. As Wenner[3] points out, this can be accomplished with the connection resistors in either of two ways. If R_H is to R_2 as R_F is to R_1, the entire right-hand expression will disappear. Also the ratios of R_G to R_2 and R_E to R_1 can be chosen to remove the term. Also, if R_H and R_F or R_G and R_E approach zero, the right-hand term will vanish leaving the desired resistance. Hamon[1] uses both effects. He makes R_F and R_H as small as possible, and matches R_E and R_G. This reduces the error as the product of the two effects so that extremely high accuracy results from moderate corrections by each technique. If the small resistors are matched too, even greater accuracy is achieved. Equation (27) can be simplified considerably by setting limits on the resistance values of the low resistance leads and on the matching of the other two leads. Then (28) can be used to find the expected accuracy:

$$\begin{gathered} R_P \approx \left(\frac{R_1R_2}{R_1 + R_2}\right)\left[1 \pm \left(\frac{1}{4}\right)\left(\frac{R_F}{R}\right)(\Delta - \delta)\right] \\ R_H \leq R_F \ll R_1,\ R_2,\ R_E,\ R_G \\ R_E \equiv R_G(1 + \delta) \\ R_1 \equiv R_2(1 + \Delta) \\ R_1 \approx R_2 \approx R. \end{gathered} \tag{28}$$

[3] F. J. Wenner, *NBS Sci. Papers*, vol. 8, 1912, p. 575.

The deviations represent the imbalance in proportional parts of the two connecting resistors and of the resistors being connected in parallel. If two resistors are connected together with a shorting bar and matched resistors on one end and a four-terminal junction on the other, as shown in Fig. 24, all of the first-order error terms can be shown as

$$R_P \approx \frac{R}{2}\left\{1 + \frac{\Delta}{2} \pm \left[\frac{M}{R} + \frac{R_F}{4R}(\Delta - \delta)\right]\right\}. \tag{29}$$

Multiple Four-Terminal Resistors in Parallel: The accuracy of the calculated parallel resistance of a group of nominally equal resistors is investigated by a perturbation technique. The effect of changing one lead resistor and one shorting bar resistor can be investigated with the circuit of Fig. 23 and (27). For the analysis, resistor R_1 is one of the resistors connected in parallel. Resistor R_2 is the parallel combination, $R/(m-1)$, of all of the rest of the resistors. Resistor R_E is one network resistor and R_G is the parallel combination of all the others. Resistor R_F is one shorting bar resistance and R_H is the parallel combination of the rest. R_H is assumed to be zero. This results in the circuit of Fig. 25 and

$$R_P = \frac{R^2(1+\Delta)}{(m-1)\left[R(1+\Delta) + \frac{R}{m-1}\right]} \times \left\{1 + \frac{R^2(1+\Delta)\left[\frac{-R_F}{R(1+\Delta)}\right]}{(m-1)\left[R_N(1+\delta) + \frac{R_N}{m-1} + R_F\right]} \times \frac{\left[\frac{R_N}{R} - \frac{R_N(1+\delta)}{R(1+\Delta)}\right]}{\left[R(1+\Delta) + \frac{R}{m-1} + \frac{\left[R_N(1+\delta) + \frac{R_N}{m-1}\right]R_F}{R_N(1+\delta) + \frac{R_N}{m-1} + R_F}\right]}\right\}. \tag{30}$$

By assuming that the shorting bar resistance is very low and that the resistor deviations are small, (30) is approximately given by

$$R_P \approx \frac{R}{m}\left[1 + \frac{\Delta}{m} + \frac{m-1}{m^2}\left(\frac{R_F}{R}\right)(\Delta - \delta)\right]. \tag{31}$$

If

$$R_F \ll R,\ R_N$$
$$1 \gg \Delta,\ \delta.$$

If each resistor contributes two errors of this magnitude then m resistors connected in parallel, as shown in

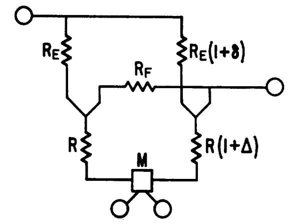


Fig. 24. Resistors in parallel.

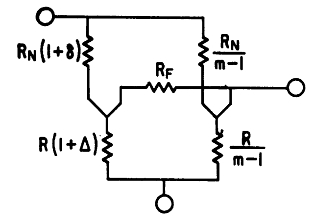


Fig. 25. Equivalent circuit for several resistors connected in parallel.

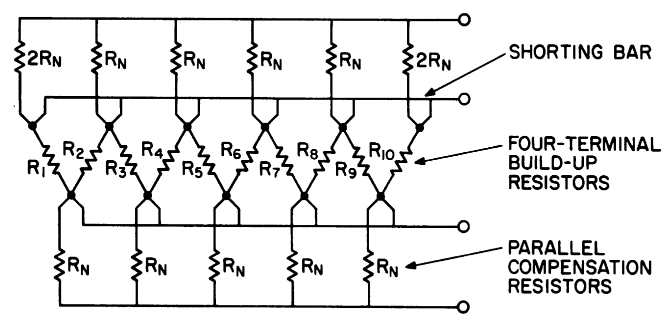


Fig. 26. Ten four-terminal resistors in parallel.

Fig. 26, would have less than the error indicated by

$$R_P \approx \frac{R}{m}\left[1 + \Delta_{av} + 2\frac{m-1}{m}\left(\frac{R_F}{R}\right)(\Delta - \delta)\right]. \tag{32}$$

R_p is the measured parallel resistance in ohms, R is the nominal value of each resistance being connected in parallel in ohms, m is the total number of resistors connected in parallel, Δ_{av} is the average deviation of resistors being connected in parallel, R_F/R is the ratio of largest shorting bar resistances to resistance of each resistor being connected in parallel, and δ is the greatest deviation of network resistors from nominal value in proportional parts. This represents a very conservative accuracy limit for several reasons. First, the imbalance of the network and main resistors cannot always have the same sign. Each case would be compared to the average of the rest so probably the worst case would be only about one-fourth that indicated. Similar reasoning would reduce the effect of the shorting bar resistance about the same amount. Matching of the shorting bar resistances causes further improvement. As a result, the expected accuracy of ten resistors connected in parallel should be at least a factor of ten better than predicted by (32). If the resistors being connected in parallel are already connected in series by four-terminal junctions which have much lower resistance than the shorting bar connection differences, the probable accuracy is reduced by about another factor of two. Thus, if (32) predicts a measurement of sufficient accuracy, it is quite reasonable to expect much better results.

By a matrix analysis of the complete circuit of Fig. 26, Page[4] arrived at essentially the same limiting error term as that shown in (32). He calculated the error in terms of the individual errors of all of the resistors. For his worst case analyses, he found that the maximum possible error in proportional parts was a product of twice the ratio of the maximum shorting bar resistance to the resistance of the resistors being paralleled, times the maximum deviation of the network resistors from their average value, times the maximum deviation of the shorting bar resistances from their average value. Using Page's notation, the constant was four instead of two, but this was caused by his using the value of two network resistors R_N in parallel for the nominal value. The other difference was in the term $(m-1)/m$. Page used the value, one, which this approaches for large m. The difference is caused by the choice of variables. Page uses difference from average while difference from the other $(m-1)$ values has been used here. In equipment of the types used by Hamon and that shown in Fig. 19, both analyses would indicate the same limit of error.

Accuracy of Series-to-Parallel Transfer: The maximum error for ten resistors connected in parallel relative to their value when connected in series can now be expressed in terms of the four-terminal junction resistance error M, the resistor value R, the network resistor unbalance δ, (for $\delta \gg \Delta$), the accuracy of the individual resistors being paralleled Δ_n, and the maximum shorting bar resistance R_F:

$$R_P = \frac{R_S}{100}\left[1 \pm 4\left(\frac{M}{R}\right) \pm 2\left(\frac{R_F}{R}\right)\delta \pm \frac{1}{10}\sum_{n=1}^{10}(\Delta_n - \Delta_{\text{av}})^2\right]. \quad (33)$$

Measuring Paralleling Errors. The values of R_F and $(\Delta-\delta)$ can be measured directly so that the measurement accuracy can be predicted. The technique for finding the ratio of shorting bar resistance to main resistor resistance is shown in Fig. 27. The technique for finding the maximum arm imbalance is shown in Fig. 28. These values substituted into (32) give a conservative estimate of the accuracy of the parallel ocnnection. Note that the values of R_N could be separately adjusted to reduce the connection error to an undetectable level.

[4] C. H. Page, *J. Research NBS*, vol. 69 C, no. 3, July–September 1965, p. 181.

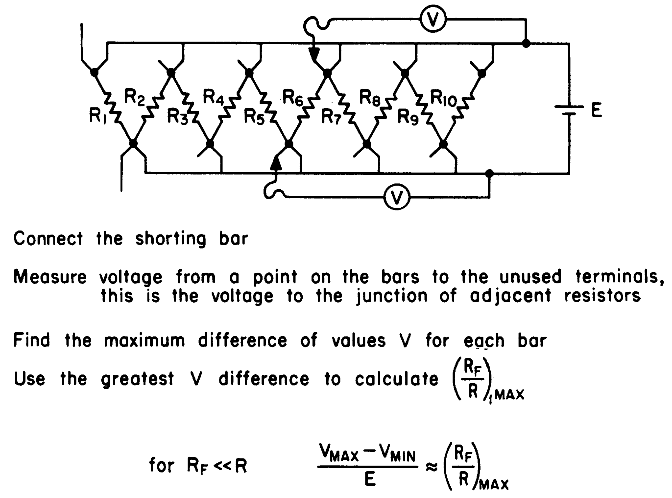


Fig. 27. Measuring the shorting bar resistance.

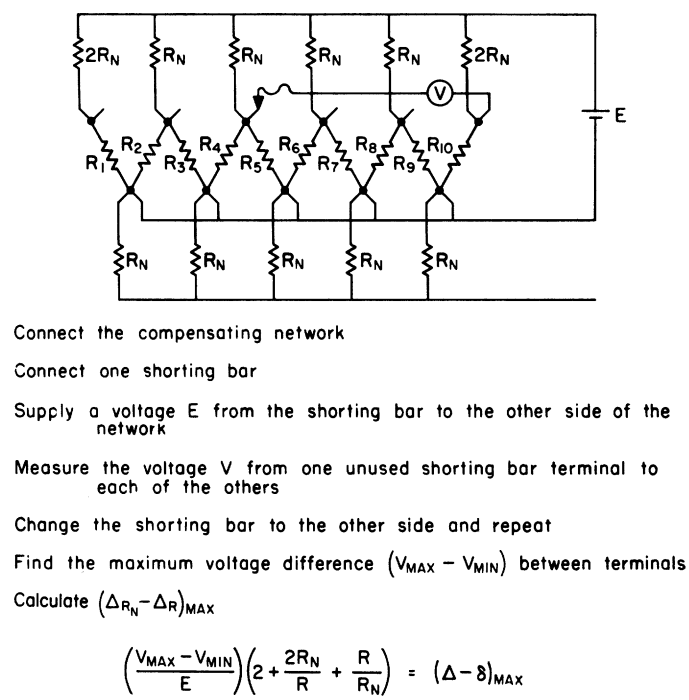


Fig. 28. Measuring the bridge imbalance.

Page also has given a set of simple measurements which permit evaluation of the errors on a specific resistance box.

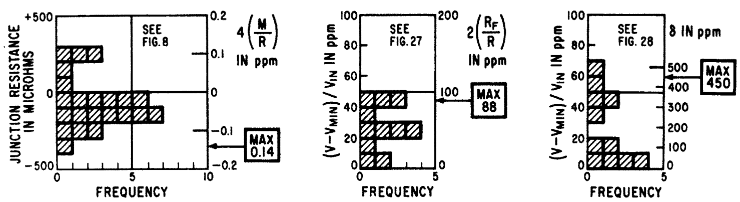
Conclusion

For the calibration of resistors of one value by a standard of a different value, the series-to-parallel transfer technique can be used with confidence at a very high accuracy level. This technique can also be modified for the calibration of precision voltage and current ratio devices. Thus, it is the base for an important segment of metrology technology. The analysis reported here was done to establish the theoretical validity of the four-

```
DELTA AVE. IS -1.8     PARTS PER MILLION
THE STANDARD DEVIATION IS  20.2771 PARTS PER MILLION
THE VARIANCE IS  411.16  PARTS IN 10 TO THE 12
```

	DELTA R
R 1	-28
R 2	-33
R 3	-13
R 4	-22
R 5	18
R 6	19
R 7	27
R 8	-2
R 9	14
R 10	2

(a)



(b)

$$R_P = \frac{R_S}{100}\left[1 \pm 4\left(\frac{M}{R}\right) \pm 2\left(\frac{R_F}{R}\right) 8 \pm \text{VARIANCE}\right]$$

$$R_P = \frac{R_S}{100}\left[1 \pm \left(0.14 + 88 \times 450 \times 10^{-6} + 412 \times 10^{-6}\right) \times 10^{-6}\right]$$

$$R_P = \frac{R_S}{100}\left[1 \pm 0.18\,\text{ppm}\right]$$

(c)

Fig. 29. Measured accuracy of one-to-one-hundred-ohm transfer using a ten ohm per step transfer standard. (a) Computer analysis of resistors showing individual resistance deviations and variance. (b) Measured junction and connection variations. (c) Worst-case accuracy calculation.

terminal junction and four-terminal connection. With the errors from these sources well below the level of measurement accuracy, the application of the series to parallel transfer technique becomes deceivingly simple. A four-terminal measurement of ten resistors in series is accomplished by comparison to a resistance standard of the same nominal value. Shorting bars are attached and a network of connecting resistors is plugged in. The freshly calibrated resistors then provide exactly one-hundredth of their series value within the predetermined accuracy limit (see Fig. 29). A series-parallel connection permits an accurate ten-to-one transfer. Comparing both the series and parallel connected values to the decade between permits a ten-to-one interchange for a simple but extremely accurate calibration of ten-to-one ratio devices.

Modern resistor technology and materials have pushed the desired resistance accuracy range well below one ppm, and these connection techniques are herein shown to be capable of such performance.

Acknowledgment

I would like to express my appreciation to M. Morgan and R. Pailthorp for their assistance with the theoretical and experimental aspects of this study, and especially to Miss H. Thomas for doing these illustrations.