Errors in the Parallel Connection of a 100:1 Series-Parallel Buildup of Four-Terminal Resistors

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Abstract—This paper deals with the problem of calculating the parallel resistance of a 100:1 series-parallel buildup of four-terminal resistors with accuracy as high as possible.

A rather complex equivalent circuit has been assumed, taking into account all the causes of significant errors. Then a complete calculation has been carried out for obtaining all the error terms expressed by means of 15 relatively synthetic formulas.

The method of solving the problem that has been developed is a new one and allows 1) rather simple calculations even if the network is complex, and 2) an immediate separation of the error terms from the main value (in fact, the result is given directly as the sum of a base value and a set of perturbations).

The method is based on applying Cohn's theorem and using certain symmetries that have been put in evidence in the equivalent circuit.

The perturbations may be considered either as corrections or as uncertainties. Very simple formulas are given for the means and the standard deviations of the errors, the perturbations being considered as uncertainties, and it being assumed that the causes of error are independent of each other and normally distributed. Anyway, these formulas may conveniently be used for initial investigation on the magnitudes of the errors before applying the more complex formulas that give the exact values of the variations.

Finally, as an example, results of computations carried out on a particular buildup box are reported.

I. Introduction

In high-accuracy metrology dc resistance buildup boxes are used, since Hamon [1] introduced them, to achieve the best accuracy in transferring the resistance calibration from one resistance level to another [2]-[4]; in other words, they are very accurate ratio devices for comparing resistance standards of different sizes.

The device consists of a set of ten equal resistors (buildup boxes with 100 resistors are being studied to obtain $10^4:1$ ratios) which, in series and in parallel connected (series connection is usually permanent), provided two resistance values that are in a 100:1 ratio. In evaluating this ratio, the hardest difficulties arise in the calculation of parallel resistance, because of the complexity of an equivalent circuit that takes account of all the network parameters that are not presumed to give negligible effects.

After the first paper by Hamon, who set the basis for the following works, noticeable contributions to the solution of the problem have been given by Riley [2] and Page [3], who developed interesting analyses of some important error terms.

The aim of this paper is to provide a theoretical basis as complete as possible for error analysis in order to obtain the highest accuracy in calculating parallel resistance R_p, by using a complex, and therefore well approximated, equivalent circuit, which makes it possible to take account of as many parameters as possible. Rather complex formulas have been obtained, many of which represent generally negligible contributions; at any rate, all of them make it possible to know with certainty which parameters are or are not negligible.

II. Equivalent Circuit

Reference has been made to the equivalent circuit of Fig. 1. The physical meanings of resistors appearing in the network are the following.

R_i	Main resistances.
R'_{ga_i}, R''_{ga_i}, R'_{gv_i}, R''_{gv_i}	Resistances of the equivalent circuits of the junctions connecting main resistors in series.
R_{al_i}	Longitudinal resistances of the two equivalent circuits representing the two bars (amperometric bars), which connect in parallel the current terminals of main resistors.
R_{vl_i}	Longitudinal resistances of the two equivalent circuits representing the two bars (voltmetric bars), which connect in parallel the potential terminals of main resistors.
R_{at_i}	Cross resistances including the cross resistances of the amperometric bars, the contact resistances at the terminals, and resistances of the equivalent circuits of the junctions.
R_{vt_i}	Cross resistances including the cross resistances of the voltmetric bars, the contact resistances, resistances of the equivalent circuits of the junctions and the matched resistances that are usually placed between the potential terminals and the voltmetric bars.

Notice that for the junctions, the equivalent circuit suggested by Searle [5], and also used by Riley [2], has been introduced. Finally, it must be noticed that port a has been placed unsymmetrically (port v is, on the contrary, symmetrical), because this arrangement is rather usual and corresponds to the disposition adopted in the buildup box on which the measurements have been made.

Manuscript received July 22, 1969.

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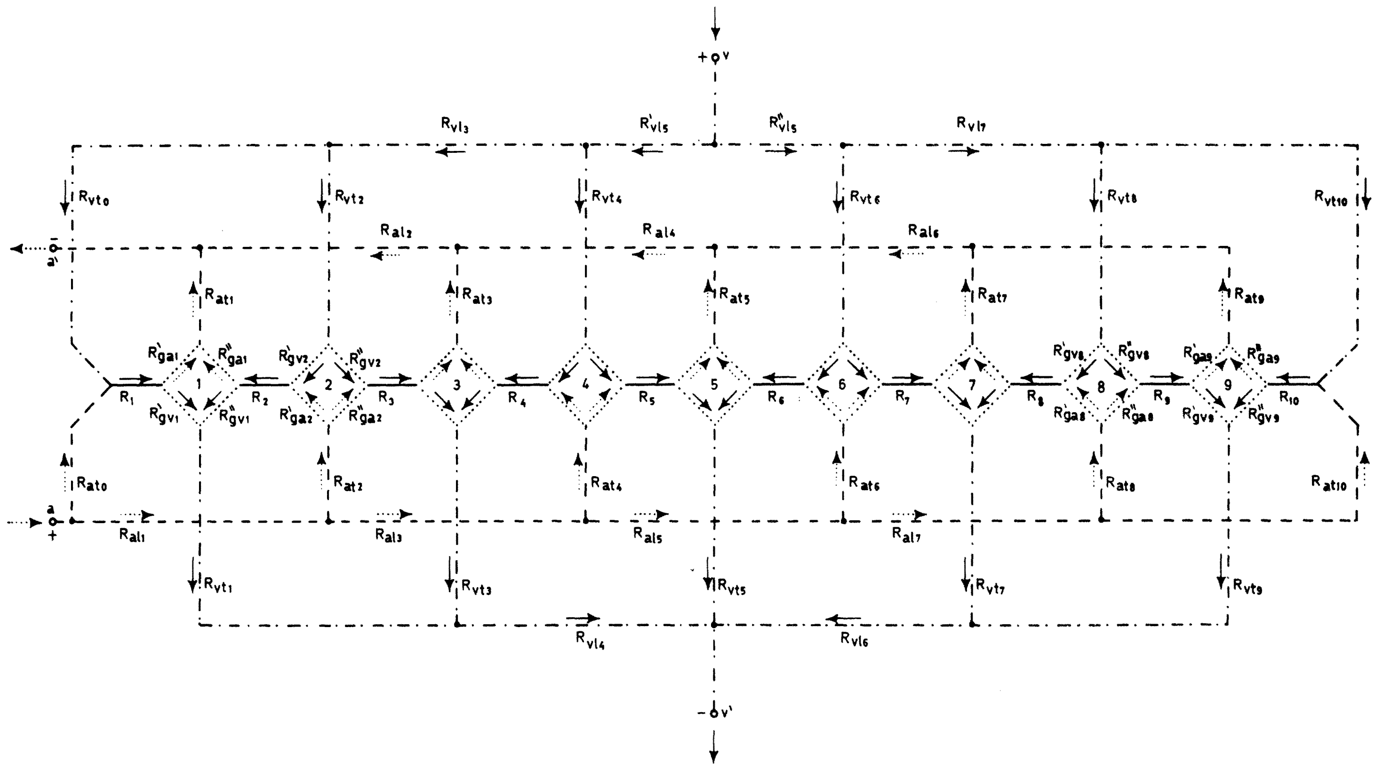

Fig. 1. Equivalent circuit representing the buildup box with the main resistors connected in parallel.

III. Calculation of Equivalent Resistance

A. Setting Out Calculation by Means of Cohn's Theorem

Equivalent resistance R_p (mutual resistance between ports a and v) has been calculated by applying Cohn's theorem [6].

This way has been preferred to other possible ones (for instance, matrix calculation), because it allows remarkable simplifications on the basis of symmetry considerations and yields an easy separation of main terms from error terms.

Calculation has been carried out considering the resistance value of each branch as the superposition of a base value and a perturbation; thereby, R_p is given as the sum of the value R_{po} presented by the base circuit and the effects due to perturbations (singular or mutual effects). Stopping the series of perturbations at the terms of second order, the expression of R_p is

$$R_p = R_{po} + \sum_\gamma \left(\frac{\partial R_p}{\partial R_\gamma}\right)_0 \Delta R_\gamma + \frac{1}{2}\sum_\gamma \sum_\delta \left(\frac{\partial^2 R_p}{\partial R_\gamma \partial R_\delta}\right) \Delta R_\gamma \Delta R_\delta \tag{1}$$

each one of the double sum

$$\sum_\gamma \sum_\delta$$

being extended to all values of γ (and δ), that is, to all the resistances of the network.

Cohn's theorem gives the following expressions of the derivatives of formula (1) (the meanings of the coefficients are shown in Fig. 2):

$$\left(\frac{\partial R_p}{\partial R_\gamma}\right)_0 = \beta_{\gamma a} \cdot \beta_{\gamma v} \tag{2}$$

$$\left(\frac{\partial^2 R_p}{\partial R_\gamma \partial R_\delta}\right)_0 = -\overset{(\circ)}{G}_{\gamma\delta}(\beta_{\delta a} \cdot \beta_{\gamma v} + \beta_{\delta v} \cdot \beta_{\gamma a}). \tag{3}$$

B. Calculation of Base Resistance

The base circuit is defined by the following assumptions (see Fig. 3).

Assumption 1: The resistances R_{al_i} and R_{vl_i} are all zero.

Assumption 2: The resistances of the same kind are all alike, therefore they are identified by a unique value: R_g, R_{at}, R_{vt}, R; R_{ato}, and R_{at10}, whose values are $2R_{at}$, make exception; similarly, the values of R_{vto} and R_{vt10} are $2R_{vt}$; it is assumed that R_g, R_{at}, R_{vt}, R are the mean values of all the resistances of the same kind. Therefore,[1]

$$\sum_i (\Delta R'_{gi} + \Delta R''_{gi}) = \sum_j \Delta R_{at_j} = \sum_\lambda \Delta R_{vt\lambda} = \sum_\mu \Delta R_\mu = 0.$$

[1] For the resistances R_{at_i} and R_{vt_i} it has been assumed:

$$R_{at} \quad \frac{1}{10}\left(\frac{R_{ato} + R_{at10}}{4} + R_{at1} + \cdots + R_{at9}\right);$$

$$R_{vt} \quad \frac{1}{10}\left(\frac{R_{vto} + R_{vt10}}{4} + R_{vt1} + \cdots + R_{vt9}\right);$$

$$\Delta R_{ato} \quad \frac{R_{ato} - 2R_{at}}{4}; \quad \Delta R_{at10} \quad \frac{R_{at10} - 2R_{at}}{4};$$

$$\Delta R_{vto} \quad \frac{R_{vto} - 2R_{vt}}{4}; \quad \Delta R_{vt10} \quad \frac{R_{vt10} - 2R_{vt}}{4}.$$

For the introduction of ΔR_{gi} and $\Delta R''_{gi}$ see footnote 3.

Assumption 3: A resistance R_g is put in series with each resistance $2R_{at}$ and $2R_{vt}$, on the side of resistances R, in order to put certain symmetries in better evidence.

For evaluating R_{po} on the base circuit it is very immaterial, because of the reciprocity theorem, to supply port a with a current I (using the terminals of port v as potential leads) or port v (using the terminals of port a as potential leads); R_{po} is given by the ratio of voltage V at potential leads to current I. Let port a be the current port. The bilateral symmetry of the network requires an analog symmetry for current and voltage distribution. Therefore, points A and C are equipotential as well as points B and D. By joining these equipotential points to each other the current and voltage distribution does not change and the new network shows a conical symmetry.[2] This symmetry involves a current and voltage distribution that is very easy to find out; in particular, the currents in resistances R_g that converge toward the R_{vt} are all zero. Equivalent resistance R_{po} is easily evaluated as

$$R_{po} = \frac{1}{10} R. \tag{4}$$

Then, R_{po} depends only on the value R; now it is possible to deduce that the conditions under which this conclusion is valid are less restrictive than those which have been imposed in the base circuit; they can be specified as the following (alternative conditions deduced from reciprocity are given in parentheses).

Condition 1) All resistances R are alike.

Condition 2) All resistances R_g are alike.[3]

Condition 3) All resistances that converge toward each terminal of port a (or v) are alike, except those that are connected to points B and D (or A and C), which have a double value.

It is worth pointing out that, if Conditions 1, 2, and 3 are verified, the structure of the network branches through which no current flows has no more importance; that is, if port a is supplied, there is no matter about the network branches that converge toward port v (or, reciprocally, toward port a, if port v is supplied). In particular, neither the equality of R_{vt} (or R_{at}), nor the presence of R_{vl} (or R_{al}), which are assumed to be zero in base network, have any importance.

The less restrictive conditions expressed in Condition 3) have not been taken into account in this paper, because of the actual conformations of buildup boxes and for reducing the complexity of calculation.

C. Calculation of Perturbation Terms

The calculation of the coefficients that appear in (2) and (3) can be made with reference to the circuit of Fig. 1, and to the base circuit conditions (namely the conditions expressed in Assumptions 1 and 2, when the circuit is being suitably supplied, according to the suggestions of Fig. 2. In the Appendix a hint concerning calculation is given; in Tables VII and VIII the coefficients and the "not zero" derivatives, are reported.[4]

In order to obtain expressions of error terms easy to handle, approximations have been introduced taking account of the sizes of the resistances, and their maximum deviations which may be present in practice. (For the physical meanings of approximation hypotheses see Section III-D.) Measurements on a 10-Ω buildup box has given an idea of such approximations.[5] Coefficients are calculated to $1 \cdot 10^{-3}$ in the worst case; accordingly, the effects of couples of resistances (e.g., R_{vto}-R_3) have not even been reported in tables, as far as they are added to the effects of other ones of the same kind (in the example, R_{vto}-R_1) that are at least 10^3 times higher. Finally, the resistances having $2R_{at}$ and $2R_{vt}$ as base values have been considered to include the resistances R_g that are in series with them (see Fig. 1).

As final results of calculations, perturbations have been reported in Table I, grouped according to the resistances to which perturbations are due. Notice that the only nonzero first derivatives are those that concern resistances R; yet their total contribution is zero because it was assumed that

$$\sum_i \Delta R_i = 0.$$

D. Physical Meaning of Hypotheses

It may be interesting to point out the physical meanings of network hypotheses under which calculations have been developed; that gives an idea of the essential technical characteristics of a buildup box, in order to apply the results of this paper. The technical characteristics that correspond to the network hypotheses are the following.

1) The main resistors (R) are connected in series by low-resistance junctions (that is, the resistances R_g are very low).

2) The connections of the current leads of main resistors are made by two shorting bars (resistances R_{al} and R_{at} are very low).

3) The connections of the voltage leads of main resistors

[2]When points A and C are joined, branches vA and vC are in parallel; then the equivalent resistance of the resulting branch has the value of the other ones that converge to the same node. When points B and D are joined an analogous fact occurs.

[3]From the previous considerations it could be deduced that the equality of all resistances of the kind R_g is not necessary; it is enough that the resistances R_{gai} and R''_{gai} (or R''_{gvi} and R''_{gvi}) in Fig. 1 are equal. As a matter of fact, because of the equivalent circuit chosen for representing the junction, the four resistances $R'_{gai}, R''_{gai}, R'_{gvi}, R''_{gvi}$ of the junction i are not independent: $R'_{gai} = R''_{gvi} = R'_{gi}$; $R''_{gai} = R'_{gai} = R''_{gi}$. Therefore, the condition that the resistances R'_{gai} and R''_{gai} (or R'_{gvi} and R''_{gvi}) be all alike coincides with the condition that all the resistances of the kind R_g be equal.

[4]Formulas (2) and (3) show the results of each derivative to be zero if either one or the other factor of the second member is zero (no coefficient is ever infinite). Then the coefficients relative to a zero derivative are not even reported.

[5]The sizes of mean values and of maximum deviations in the box on which the measures are carried out follow:

Resistance	*Mean* (Ω)	*Maximum Deviation* (Ω)
R_g	$5 \cdot 10^{-6}$	$4 \cdot 10^{-7}$
R_{al}	10^{-5}	$5 \cdot 10^{-6}$
R_{vl}	$5 \cdot 10^{-5}$	$2 \cdot 10^{-6}$
R_{at}	10^{-4}	$4 \cdot 10^{-5}$
R_{vt}	1	10^{-3}
R	10	10^{-4}

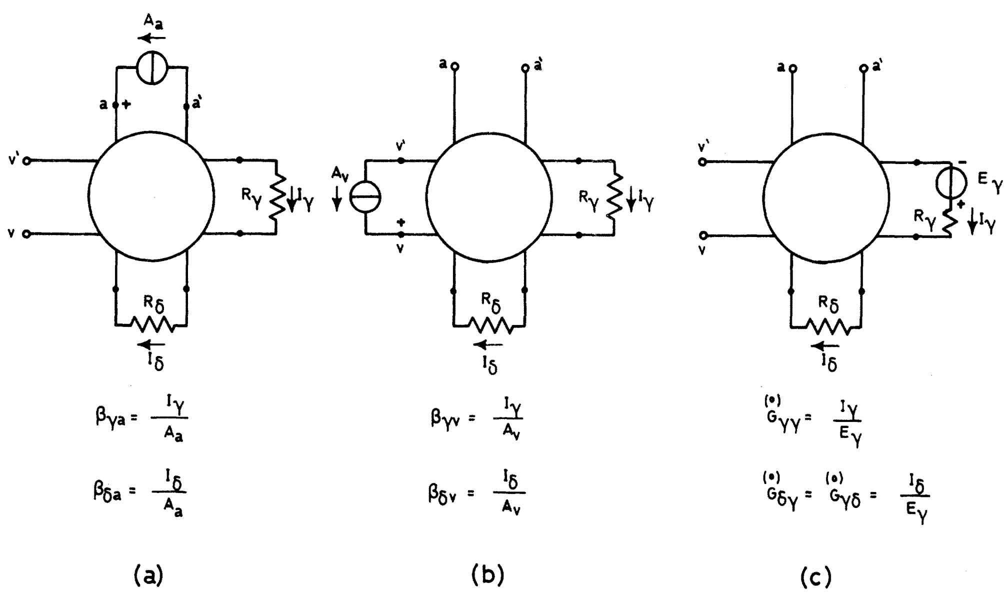

Fig. 2. Explanation of the meanings of coefficients introduced for applying Cohn's theorem.

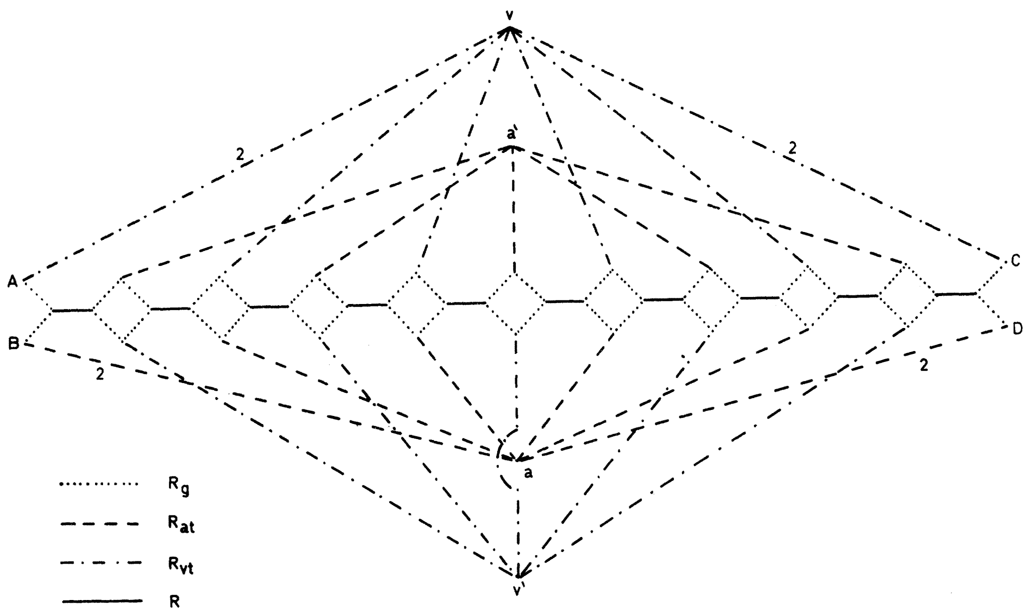

Fig. 3. Base circuit introduced for applying Cohn's theorem.

are made by additional matched resistors (R_{vt})[6] of the same sizes, joined at each voltage terminal of the box (the terminals of port v) by two shorting bars (resistances R_{vl} are very low).

4) The leakage resistances are negligible; their importance depends obviously on the sizes of main resistances R (and on the accuracy aimed for).

These conditions could seem restrictive, but in practice they can be related to the buildup boxes that now are dealt with (except the boxes with high main resistances, because of leakage resistance paths).

[6]The usefulness of introducing matched resistors in paralleling potential terminals (and not in paralleling current terminals) for reducing error terms was first noted by Hamon, then by Riley and Page. As a consequence of this introduction $R_{vt} \gg R_{at}$, as we have assumed in this paper.

IV. Interpretation of Perturbation Terms as Uncertainties

The formulas of Table I give the variations, to be added to R_{po} [see (1)], which can be calculated if the resistances that they depend on are known. If they are not known (for instance, because they depend on contact resistances that are not negligible, as may be the case of resistances R_{ati}), they can be assumed to be independent random variables with normal probability distribution. The variations given by the formulas are random variables too, the mean values and standard deviations of which can be found as functions of the mean values and standard deviations of resistances.

Assuming that resistances of the same kind (e.g., R_{vt}) belong to the same population, the parameters of Table II are introduced and the results of Table III are obtained.

TABLE I

Formula	Kinds of Resistances Referred to	Variations	Index Field of Variation	Notes
1	R_g	$-\frac{2.5\cdot10^{-3}}{R_g}\sum_i(\Delta R'_{gi}-\Delta R''_{gi})^2$	$1\le i\le 9$	
2	$R_g,\ R_{al}$	$\frac{2.5\cdot10^{-3}}{R}\sum_i(10-i)R_{al_i}(\Delta R''_{gi}-\Delta R'_{gi})$	$1\le i\le 7$	
3	$R_g,\ R_{vl}$	$\frac{5\cdot10^{-4}}{R_{vt}}\sum_i i\Big\{-\sum_k i\Big[R_{vl_i}(\Delta R'_{ga(i+2k-1)}+\Delta R''_{ga(i+2k-1)})+R_{vl(10-i)}(\Delta R'_{ga(11-i-2k)}+\Delta R''_{ga(11-i-2k)})\Big]+\sum_h(10-i)\Big[R_{vl_i}(\Delta R'_{ga(i-2h+1)}+\Delta R''_{ga(i-2h+1)})+R_{vl(10-i)}\cdot(\Delta R'_{ga(9-i+2h)}+\Delta R''_{ga(9-i+2h)})\Big]\Big\}$	$3\le i\le 5$ $4\le i+2k-1\le 9;\ k>0$ $1\le i-2h+1\le 4;\ h>0$	for $i=5$: $\begin{cases}R_{vl_i}=R'_{vl5}\\R_{vl(10-i)}=R''_{vl5}\end{cases}$
4	$R_g,\ R_{at}$	$\frac{2\cdot10^{-3}}{R_{vt}}\Big[4\sum_i\Delta R_{at_i}(\Delta R'_{gi}+\Delta R''_{gi})-\sum_j\sum_k\Delta R_{at_j}(\Delta R'_{g(j\pm2k)}+\Delta R''_{g(j\pm2k)})\Big]$	$1\le i\le 9$ $0\le j\le 10$ $1\le j\pm2k\le 9;\ k>0$	
5	$R_g,\ R_{vt}$	$\frac{2\cdot10^{-3}}{R_{vt}}\Big[4\sum_i\Delta R_{vt_i}(\Delta R'_{gi}+\Delta R''_{gi})-\sum_j\sum_k\Delta R_{vt_j}(\Delta R'_{g(j\pm2k)}+\Delta R''_{g(j\pm2k)})\Big]$	$1\le i\le 9$ $0\le j\le 10$ $1\le j\pm2k\le 9;\ k>0$	
6	$R_g,\ R$	$\frac{10^{-3}}{R}\Big\{-4\sum_i(\Delta R'_{gi}+\Delta R''_{gi})(\Delta R_i+\Delta R_{i+1})+\sum_j\sum_k\Big[\Delta R_j(\Delta R'_{g(j+k)}+\Delta R''_{g(j+k)})+\Delta R_{(11-j)}(\Delta R'_{g(10-j-k)}+\Delta R''_{g(10-j-k)})\Big]\Big\}$	$1\le i\le 9$ $1\le j\le 8$ $2\le j+k\le 9;\ k>0$	
7	$R_{al},\ R_{vl}$	$\frac{5\cdot10^{-4}}{R_{vt}}\Big\{\sum_i i^2(10-i)\Big[-(10-i)R_{al_i}R_{vl_i}+iR_{al(10-i)}R_{vl(10-i)}\Big]-\sum_j\sum_k\Big[j(10-j)\cdot(j+2k)(10-j-2k)R_{al_j}R_{vl(j+2k)}+(2+j)^2(8-j-2k)^2R_{al(8-j)}R_{vl(8-j-2k)}\Big]+\sum_r\sum_h r\cdot(10-r)(10-r-2h)^2R_{al_r}R_{vl(r+2h)}+225R_{al_5}R''_{vl5}\Big\}$	$3\le i\le 5$ $1\le j\le 3$ $3\le j+2k\le 5;\ k>0$ $1\le r\le 5$ $5\le r+2h\le 7;\ h>0$	for $i=5$: $\begin{cases}R_{vl_i}=R'_{vl5}\\R_{vl(10-i)}=R''_{vl5}\end{cases}$ for $j+2k=5$: $R_{vl(j+2k)}=R'_{vl5}$ for $8-j-2k=5$: $R_{vl(8-j-2k)}=R'_{vl5}$ for $r+2h=5$: $R_{vl(r+2h)}=R''_{vl5}$
8	$R_{al},\ R_{vt}$	$\frac{2\cdot10^{-3}}{R_{vt}}\sum_i(10-i)R_{al_i}\Big[\sum_k i\Delta R_{vt(i+2k-1)}-\sum_h(10-i)\Delta R_{vt(i-2h+1)}\Big]$	$1\le i\le 7$ $2\le i+2k-1\le 10;\ k>0$ $0\le i-2h+1\le 6;\ h>0$	
9	$R_{al},\ R$	$\frac{10^{-3}}{R}\sum_i(10-i)R_{al_i}\Big[-\sum_k i\Delta R_{(i+k)}+\sum_h(10-i)\Delta R_{(i-h)}\Big]$	$1\le i\le 7$ $2\le i+k\le 10;\ k>0$ $1\le i-h\le 7;\ h\ge 0$	
10	$R_{vl},\ R_{at}$	$\frac{2\cdot10^{-3}}{R_{vt}}\sum_i i\Big[-\sum_k i(R_{vl_i}\Delta R_{at(i+2k-1)}+R_{vl(10-i)}\Delta R_{at(11-i-2k)})+\sum_h(10-i)(R_{vl_i}\cdot\Delta R_{at(i-2h+1)}+R_{vl(10-i)}\Delta R_{at(9-i+2h)})\Big]$	$3\le i\le 5$ $4\le i+2k-1\le 10;\ k>0$ $0\le i-2h+1\le 4;\ h>0$	
11	$R_{vl},\ R$	$\frac{5\cdot10^{-4}R_{at}}{R_{vt}R}\sum_i i\Big[-(10-i)R_{vl_i}(\Delta R_i+\sum_k\Delta R_{(i-k)})+\sum_h iR_{vl_i}\Delta R_{(i+h)}+i\cdot R_{vl(10-i)}(\Delta R_{(10-i)}+\sum_\lambda\Delta R_{(10-i-\lambda)})-\sum_\mu(10-i)R_{vl(10-i)}\Delta R_{(10-i+\mu)}\Big]$	$3\le i\le 5$ $1\le i-k\le 4;\ k>0$ $4\le i+h\le 10;\ h>0$ $1\le 10-i-\lambda\le 6;\ \lambda>0$ $6\le 10-i+\mu\le 10;\ \mu>0$	for $i=5$: $\begin{cases}R_{vl_i}=R'_{vl5}\\R_{vl(10-i)}=R''_{vl5}\end{cases}$
12	$R_{at},\ R_{vt}$	$\frac{4\cdot10^{-2}}{R_{vt}}\Big[\sum_i\Big(\Delta R_{at_i}\Delta R_{vt_i}+0.2\sum_k\Delta R_{at_i}\Delta R_{vt[i\pm(2k-1)]}\Big)+\Delta R_{at_0}\Delta R_{vt_0}+\Delta R_{at_{10}}\Delta R_{vt_{10}}\Big]$	$0\le i\le 10$ $0\le i\pm(2k-1)\le 10;\ k>0$	
13	$R_{at},\ R$	$-\frac{2\cdot10^{-2}}{R}\Big[\sum_i\Delta R_i(\Delta R_{at_i}+\Delta R_{at(i-1)})+\Delta R_{at_0}\Delta R_1+\Delta R_{at_{10}}\Delta R_{10}\Big]$	$1\le i\le 10$	
14	$R_{vt},\ R$	$-\frac{2\cdot10^{-2}}{R}\Big[\sum_i\Delta R_i(\Delta R_{vt_i}+\Delta R_{vt(i-1)})+\Delta R_{vt_0}\Delta R_1+\Delta R_{vt_{10}}\Delta R_{10}\Big]$	$1\le i\le 10$	
15	R	$-\frac{10^{-2}}{R}\sum_i(\Delta R_i)^2$	$1\le i\le 10$	

TABLE II

Resistance	Value Assumed in the Base Circuit	Value to Which Reference is Made for Calculating Variations	Mean Value Assumed	Standard Deviation Assumed
R_g[a]	R_g	$\Delta R'_{gi}$; $\Delta R''_{gi}$	0	σ_{Rg}
R_{al}	0	R_{ali}	R_{al}	σ_{Ral}
R_{vl}	0	R_{vli}	R_{vl}	σ_{Rvl}
R_{at}	R_{at}	ΔR_{ati}	0	σ_{Rat}
R_{vt}	R_{vt}	ΔR_{vti}	0	σ_{Rvt}
R	R	ΔR_i	0	σ_R

[a] $\Delta R'_{gi}$ and $\Delta R''_{gi}$ have been assumed to belong to the same population.

TABLE III

Formula	Kinds of Resistances Referred to	Mean Value	Standard Deviation
1	R_g	$-\frac{4.5\cdot 10^{-2}\cdot\sigma_{Rg}^2}{R_g}$	$\frac{2\cdot 10^{-2}\cdot\sigma_{Rg}^2}{R_g}$
2	R_g, R_{al}	0	$\frac{6\cdot 10^{-2}}{R}\sigma_{Rg}\sqrt{\sigma_{Ral}^2+R_{al}^2}$
3	R_g, R_{vl}	0	$\frac{7\cdot 10^{-2}}{R_{vt}}\sigma_{Rg}\sqrt{\sigma_{Rvl}^2+R_{vl}^2}$
4	R_g, R_{at}	0	$\frac{4\cdot 10^{-2}\sigma_{Rg}\cdot\sigma_{Rat}}{R_{vt}}$
5	R_g, R_{vt}	0	$\frac{4\cdot 10^{-2}\sigma_{Rg}\cdot\sigma_{Rat}}{R_{vt}}$
6	R_g, R	0	$\frac{3\cdot 10^{-2}\sigma_{Rg}\cdot\sigma_R}{R}$
7	R_{al}, R_{vl}	$-\frac{2.2\cdot 10^{-1}R_{al}\cdot R_{vl}}{R_{vt}}$	$\frac{8.5\cdot 10^{-1}}{R_{vt}}\sqrt{(\sigma_{Ral}^2+R_{al}^2)(\sigma_{Rvl}^2+R_{vl}^2)+R_{al}^2R_{vl}^2}$
8	R_{al}, R_{vt}	0	$\frac{3.5\cdot 10^{-1}}{R_{vt}}\sigma_{Rvt}\sqrt{\sigma_{Ral}^2+R_{al}^2}$
9	R_{al}, R	0	$\frac{2\cdot 10^{-1}}{R}\sigma_R\sqrt{\sigma_{Ral}^2+R_{al}^2}$
10	R_{vl}, R_{at}	0	$\frac{2.5\cdot 10^{-1}}{R_{vt}}\sigma_{Rat}\sqrt{\sigma_{Rvl}^2+R_{vl}^2}$
11	R_{vl}, R	0	$\frac{7.7\cdot 10^{-2}R_{at}}{R_{vt}\cdot R}\sigma_R\sqrt{\sigma_{Rvl}^2+R_{vl}^2}$
12	R_{at}, R_{vt}	0	$\frac{1.5\cdot 10^{-1}\sigma_{Rat}\cdot\sigma_{Rvt}}{R_{vt}}$
13	R_{at}, R	0	$\frac{9.5\cdot 10^{-2}\sigma_{Rat}\cdot\sigma_R}{R}$
14	R_{vt}, R	0	$\frac{9.5\cdot 10^{-2}\sigma_{Rvt}\cdot\sigma_R}{R_{vt}\cdot R}$
15	R	$-\frac{10^{-1}\sigma_R^2}{R}$	$\frac{10^{-1}\sigma_R^2}{R}$

TABLE IV

i	$\Delta R'_{gi}$ ($\mu\Omega$)	$\Delta R''_{gi}$ ($\mu\Omega$)	R_{ali} ($\mu\Omega$)	R_{vli} ($\mu\Omega$)	ΔR_{ati}[a] ($\mu\Omega$)	ΔR_{vti}[a] (mΩ)	ΔR_i ($\mu\Omega$)
0	–	–	–	–	-4	-0.6	–
1	-0.29	-0.49	18	–	-26	0.7	-9
2	0.21	0.01	16	–	1	–	102
3	-0.09	-0.09	10	70	-10	-0.2	59
4	0.61	0.41	12	68	15	0.4	10
5 {5′	-0.49	-0.49	18	34	22	0.0	6
5″				33			
6	-0.49	-0.29	16	69	-2	-0.4	-44
7	0.31	0.31	18	68	-15	0.2	-57
8	-0.09	-0.09	–	–	-35	0.2	27
9	0.61	0.41	–	–	23	0.2	-1
10	–	–	–	–	31	-0.7	-94

	R_g	R_{al}	R_{vl}	R_{at}[a]	R_{vt}[a]	R
Mean (Ω)	$50 \cdot 10^{-8}$	$15.3 \cdot 10^{-6}$	$68 \cdot 10^{-6}$	$236 \cdot 10^{-6}$	1.0006	9.999437
Standard deviation (Ω)	$4 \cdot 10^{-7}$	$3 \cdot 10^{-6}$	$1.4 \cdot 10^{-6}$	$21 \cdot 10^{-6}$	$4 \cdot 10^{-4}$	$57 \cdot 10^{-6}$

[a]See footnote 1.

TABLE V

Corresponding Formula of Table I	Kinds of Resistances Referred to	Variation (Ω)
1	R_g	$-1 \cdot 10^{-9}$
2	R_g, R_{al}	$-1.5 \cdot 10^{-14}$
3	R_g, R_{vl}	$5.7 \cdot 10^{-13}$
4	R_g, R_{at}	$3.7 \cdot 10^{-13}$
5	R_g, R_{vt}	$6.5 \cdot 10^{-12}$
6	R_g, R	$1.8 \cdot 10^{-14}$
7	R_{al}, R_{vl}	$-1.8 \cdot 10^{-10}$
8	R_{al}, R_{vt}	$-7 \cdot 10^{-10}$
9	R_{al}, R	$6 \cdot 10^{-11}$
10	R_{vl}, R_{at}	$-1.7 \cdot 10^{-10}$
11	R_{vl}, R	$-3.4 \cdot 10^{-15}$
12	R_{at}, R_{vt}	$-2 \cdot 10^{-9}$
13	R_{at}, R	$2.3 \cdot 10^{-11}$
14	R_{vt}, R	$-1.2 \cdot 10^{-13}$
15	R	$-2.9 \cdot 10^{-11}$

TABLE VI

Corresponding Formula of Tables I and III	Kinds of Resistances Referred to	Values Calculated by Means of the Formulas of Table III: Mean (Ω)	Standard Deviation (Ω)	Variation Calculated by Means of the Formulas of Table I (Ω)
1	R_g	$-1.5 \cdot 10^{-8}$	$3 \cdot 10^{-9}$	$-1 \cdot 10^{-9}$
2	R_g, R_{al}	0	$3.4 \cdot 10^{-14}$	$-1.5 \cdot 10^{-14}$
3	R_g, R_{vl}	0	$1.8 \cdot 10^{-12}$	$5.7 \cdot 10^{-13}$
4	R_g, R_{at}	0	$3 \cdot 10^{-13}$	$3.7 \cdot 10^{-13}$
5	R_g, R_{vt}	0	$1.9 \cdot 10^{-12}$	$6.5 \cdot 10^{-12}$
6	R_g, R	0	$6.5 \cdot 10^{-14}$	$1.8 \cdot 10^{-14}$
7	R_{al}, R_{vl}	$-2.2 \cdot 10^{-10}$	$1.2 \cdot 10^{-9}$	$-1.8 \cdot 10^{-10}$
8	R_{al}, R_{vt}	0	$7.5 \cdot 10^{-10}$	$-7 \cdot 10^{-10}$
9	R_{al}, R	0	$8.5 \cdot 10^{-12}$	$6 \cdot 10^{-11}$
10	R_{vl}, R_{at}	0	$1.4 \cdot 10^{-10}$	$-1.7 \cdot 10^{-10}$
11	R_{vl}, R	0	$7.2 \cdot 10^{-15}$	$-3.4 \cdot 10^{-15}$
12	R_{at}, R_{vt}	0	$1 \cdot 10^{-9}$	$-2 \cdot 10^{-9}$
13	R_{at}, R	0	$1.2 \cdot 10^{-10}$	$2.3 \cdot 10^{-11}$
14	R_{vt}, R	0	$6.6 \cdot 10^{-14}$	$-1.2 \cdot 10^{-13}$
15	R	$-3.2 \cdot 10^{-11}$	$3.2 \cdot 10^{-11}$	$-2.9 \cdot 10^{-11}$

TABLE VII

Kind of Resistance Referred to (R)	Field of Variation of Index i	Number of Coefficients	Coefficients of $\beta_{\gamma a}$: Name	Value	Coefficients of $\beta_{\gamma v}$: Name	Value	Derivative of First Order $(\partial R_p/\partial R)_0$
R_{ga}[a]	1 - 9	18	$\beta_{gai\text{-}a}$	0.1	$\beta_{gai\text{-}v}$	0	0
R_{gv}[a]	1 - 9	18	$\beta_{gvi\text{-}a}$	0	$\beta_{gvi\text{-}v}$	0.1	0
R_{al}	1 - 7	7	$\beta_{ali\text{-}a}$	$0.1(10-i)$	$\beta_{ali\text{-}v}$	0	0
R_{vl}[b]	3 - 5′	3	$\beta_{vli\text{-}a}$	0	$\beta_{vli\text{-}v}$	$0.1 \cdot i$	0
	5″- 7	3		0		$0.1(10-i)$	0
R_{at}	1 - 9	9	$\beta_{ati\text{-}a}$	0.2	$\beta_{ati\text{-}v}$	0	0
	0 -10	2		0.1		0	0
R_{vt}	1 - 9	9	$\beta_{vti\text{-}a}$	0	$\beta_{vti\text{-}v}$	0.2	0
	0 -10	2		0		0.1	0
R	1 -10	10	$\beta_{i\text{-}a}$	0.1	$\beta_{i\text{-}v}$	0.1	0.01

[a]According to Fig. 1 each value assumed by index i corresponds to a couple of resistances of the kind R_{ga}(R'_{gai} and R''_{gai}) and of the kind R_{gv}(R'_{gvi} and R''_{gvi}); there is a unique value of the coefficient corresponding to each couple of resistances.

[b]With reference to Fig. 1 it may be remarked that two values of R_{vli} exist when $i = 5$; namely R'_{v15} and R''_{v15}. The fact has been put in evidence introducing $i = 5'$ (which corresponds to R'_{v15}) and $i = 5''$ (which corresponds to R''_{v15}).

TABLE VIII

Corresponding Formula of Table I	Kinds of Resistances Referred to $(R_\gamma R_\delta)$	Coefficients Name	Coefficients Value	Index Field of Variation	Number of Coefficients	Second-Order Derivatives $\left(\frac{\partial^2 R_p}{\partial R_\gamma \partial R_\delta}\right)_0$
1	R'_{ga}, R'_{gv}	$\overset{(0)}{G}_{g'ai-g'vi}$	$-\frac{1}{4R_g}$	$1 \leq i \leq 9$	9	$\frac{10^{-2}}{4R_g}$
	R''_{ga}, R''_{gv}	$\overset{(0)}{G}_{g''ai-g''vi}$	»	»	»	»
	R'_{ga}, R''_{gv}	$\overset{(0)}{G}_{g'ai-g''vi}$	$\frac{1}{4R_g}$	»	»	$-\frac{10^{-2}}{4R_g}$
	R''_{ga}, R'_{gv}	$\overset{(0)}{G}_{g''ai-g'vi}$	»	»	»	»
2	R'_{gv}, R_{al}	$\overset{(0)}{G}_{g'vi-ali}$	$-\frac{1}{4R}$	$1 \leq i \leq 7$	7	$\frac{(10-i)10^{-2}}{4R}$
	R''_{gv}, R_{al}	$\overset{(0)}{G}_{g''vi-ali}$	$\frac{1}{4R}$	»	»	$-\frac{(10-i)10^{-2}}{4R}$
3 (*)	R'_{ga}, R_{vt}	$\overset{(0)}{G}_{g'a(i+2k-1)-vti}$	$\frac{i}{20R_{vt}}$	$3 \leq i \leq 5'$; $4 \leq i+2k-1 \leq 9$; $k>0$	8	$-\frac{i^2 10^{-3}}{2R_{vt}}$
		»	$-\frac{i}{20R_{vt}}$	$5'' \leq i \leq 7$; $6 \leq i+2k-1 \leq 9$; »	5	$\frac{i(10-i)10^{-3}}{2R_{vt}}$
		$\overset{(0)}{G}_{g'a(i-2k+1)-vti}$	$-\frac{10-i}{20R_{vt}}$	$3 \leq i \leq 5'$; $1 \leq i-2k+1 \leq 4$; »	5	»
		»	$\frac{10-i}{20R_{vt}}$	$5'' \leq i \leq 7$; $1 \leq i-2k+1 \leq 6$; »	8	$-\frac{(10-i)^2 10^{-3}}{2R_{vt}}$
	R''_{ga}, R_{vt}	Coefficients coincide exactly with the coefficients corresponding to R'$_{ga}$, R$_{vl}$ (by substituting R $_{ga}$ with R''$_{ga}$).				
4	R'_{gv}, R_{at}	$\overset{(0)}{G}_{g'vi-ati}$	$-\frac{2}{5R_{vt}}$	$1 \leq i \leq 9$	9	$\frac{8 \cdot 10^{-3}}{R_{vt}}$
		$\overset{(0)}{G}_{g'v(i\pm 2k)-ati}$	$\frac{1}{10R_{vt}}$	» ; $1 \leq i \pm 2k \leq 9$; $k>0$	32	$-\frac{2 \cdot 10^{-3}}{R_{vt}}$
		»	$\frac{1}{20R_{vt}}$	$i=0;10$; $1 \leq i \pm 2k \leq 9$; $k>0$	8	$-\frac{10^{-3}}{3R_{vt}}$
	R''_{gv}, R_{at}	Coefficients coincide exactly with the coefficients corresponding to R'$_{gv}$, R$_{at}$ (by substituting R'$_{gv}$ with R''$_{gv}$).				
5	R'_{ga}, R_{vt}	Coefficients coincide exactly with the coefficients corresponding to R'$_{gv}$, R$_{at}$ (by substituting R'$_{gv}$ with R'$_{ga}$ and R$_{at}$ with R$_{vt}$).				
	R''_{ga}, R_{vt}	Coefficients coincide exactly with the coefficients corresponding to R'$_{gv}$, R$_{at}$ (by substituting R'$_{gv}$ with R''$_{ga}$ and R$_{at}$ with R$_{vt}$).				

V. Comparison With the Results of Other Authors

As has been mentioned in Section I calculations of errors in series-parallel buildup boxes have been developed by Riley [2] and Page [3]. It must be remarked that they have deduced only the effects of few causes of errors (the main ones, indeed), without saying anything about the others; formulas corresponding to 5 among the 15 of this paper have been calculated complexly by Riley and Page. Moreover, the calculations of the means and of the standard deviations of errors have not been yet carried out.

As for the comparison of the corresponding formulas, it must be noticed that it is not always possible because of the different calculation techniques and the different equivalent circuits that have been assumed. Yet, at least, the structures of formulas and the sizes of errors can be compared. This comparison has been carried out in the Appendix.

VI. Technical Remarks on the Construction of a Buildup Box

It is obvious to see what the characteristics of the base circuit are aimed at, since they would make it possible to obtain R_p by the simple formula (4); technical meanings of network hypotheses made on the base circuit have been explained in Section III-D. The results reported in Tables I and III give some suggestions, at least, as confirmations of intuition, concerning the approach to be followed in order that unavoidable defects yield perturbations as low as possible.

R_g: Formula 1 of Table I shows that equality of R'_{gi} and R''_{gi} of each junction is requested independently of the values assumed in other junctions; the physical meaning of this condition has been explained quite well by Riley [2]. Formula 1 of Table III leads to considerations on the size of R_g; minimization of $(\sigma_{R_g}/R_g)\ \sigma_{R_g}$ is requested as a compromise be-

TABLE VIII (*Cont'd.*).

Corresponding Formula of Table I	Kinds of Resistances Referred to $(R_\gamma R_\delta)$	Coefficients Name	Coefficients Value	Index Field of Variation	Number of Coefficients	Second-Order Derivatives $\left(\frac{\partial^2 R_p}{\partial R_\gamma \partial R_\delta}\right)_0$
6	R'_{ga}, R	$G^{(0)}_{g'ai-i}$	$\frac{13}{20R}$	$1 \le i \le 9$	9	$-\frac{13 \cdot 10^{-3}}{2 \cdot R}$
		$G^{(0)}_{g'ai-(i+1)}$	$\frac{3}{20R}$	〃	〃	$-\frac{3 \cdot 10^{-3}}{2R}$
		$G^{(0)}_{g'a(i+k)-i}$	$-\frac{1}{10R}$	$1 \le i \le 8$; $2 \le i+k \le 9$; $k>0$	36	$\frac{10^{-3}}{R}$
		$G^{(0)}_{g'a(i-k-1)-i}$	〃	$3 \le i \le 10$; $1 \le i-k-1 \le 8$; 〃	〃	〃
	R''_{ga}, R	$G^{(0)}_{g''ai-i}$	$\frac{3}{20R}$	$1 \le i \le 9$	9	$-\frac{3 \cdot 10^{-3}}{2R}$
		$G^{(0)}_{g''ai-(i+1)}$	$\frac{13}{20R}$	〃	〃	$-\frac{13 \cdot 10^{-3}}{2R}$
		$G^{(0)}_{g''a(i+k)-i}$	$-\frac{1}{10R}$	$1 \le i \le 8$; $2 \le i+k \le 9$; $k>0$	36	$\frac{10^{-3}}{R}$
		$G^{(0)}_{g''a(i-k-1)-i}$	〃	$3 \le i \le 10$; $1 \le i-k-1 \le 8$; 〃	〃	〃
	R'_{gv}, R	$G^{(0)}_{g'vi-i}$	$\frac{1}{4R}$	$1 \le i \le 9$	9	$-\frac{10^{-2}}{4R}$
		$G^{(0)}_{g'vi-(i+1)}$	$-\frac{1}{4R}$	〃	〃	$\frac{10^{-2}}{4R}$
	R''_{gv}, R	$G^{(0)}_{g''vi-i}$	〃	〃	〃	〃
		$G^{(0)}_{g''vi-(i+1)}$	$\frac{1}{4R}$	〃	〃	$-\frac{10^{-2}}{4R}$
7 (*)	R_{al}, R_{vl}	$G^{(0)}_{ali-vli}$	$\frac{i(10-i)}{20R_{vt}}$	$3 \le i \le 5'$	3	$-\frac{i^2(10-i)^2 10^{-3}}{2R_{vt}}$
		〃	$-\frac{i(10-i)}{20R_{vt}}$	$5'' \le i \le 7$	〃	$\frac{i(10-i)^3 10^{-3}}{2R_{vt}}$
		$G^{(0)}_{ali-vl(i+2k)}$	$\frac{i(10-i-2k)}{20R_{vt}}$	$1 \le i \le 3$; $3 \le i+2k \le 5'$; $k>0$	4	$-\frac{i(10-i-2k)(10-i)(i+2k)10^{-3}}{2R_{vt}}$
		〃	$-\frac{i(10-i-2k)}{20R_{vt}}$	$1 \le i \le 5$; $5'' \le i+2k \le 7$; 〃	7	$\frac{i(10-i-2k)^2(10-i)10^{-3}}{2R_{vt}}$
		$G^{(0)}_{ali-vl(i-2k)}$	$\frac{(10-i)(i-2k)}{20R_{vt}}$	$5 \le i \le 7$; $3 \le i-2k \le 5$; 〃	4	$-\frac{(10-i)^2(i-2k)^2 10^{-3}}{2R_{vt}}$
		〃	$-\frac{(10-i)(i-2k)}{20R_{vt}}$	$i=7$; $i-2k=5''$	1	$\frac{(10-i)^2(i-2k)(10-i+2k)10^{-3}}{2R_{vt}}$
8	R_{al}, R_{vt}	$G^{(0)}_{ali-vt(i+2k-1)}$	$-\frac{i}{10R_{vt}}$	$1 \le i \le 7$; $2 \le i+2k-1 \le 9$; $k>0$	19	$\frac{2i(10-i)10^{-3}}{R_{vt}}$
		〃	$-\frac{i}{20R_{vt}}$	〃; $i+2k-1=10$	4	$\frac{i(10-i)10^{-3}}{2R_{vt}}$
		$G^{(0)}_{ali-vt(i-2k+1)}$	$\frac{10-i}{10R_{vt}}$	$2 \le i \le 7$; $1 \le i-2k+1 \le 6$; $k>0$	12	$-\frac{2(10-i)^2 10^{-3}}{R_{vt}}$
		〃	$\frac{10-i}{20R_{vt}}$	$1 \le i \le 7$; $i-2k+1=0$	4	$-\frac{(10-i)^2 10^{-3}}{2R_{vt}}$

tween the minimization of (σ_{R_g}/R_g) (which corresponds to a size of R_g not too small) and of σ_{R_g} (which corresponds to a low value of R_g). Notice that formulas 2-6 of Table III require σ_{R_g} to be low; an overall compromise is therefore requested. Generally, it can be said that contributions of formulas 2-6 are negligible compared to that of formula 1; consequently, R_g might be requested not to be as low as possible.

R_{al}: Formulas 2, 7-9 show the suitability of making R_{al} as low as possible also for minimizing $\sigma_{R_{al}}$.

R_{vl}: The analysis of formulas 3, 7, and 11 yields a conclusion similar to that obtained for R_{al} (low value).

R_{at}: Formulas 4, 10, 12, and 13 suggest minimizing the value of $\sigma_{R_{at}}$, that is R_{at} (a low value of R_{at} is an assumption in the calculations of this paper).

TABLE VIII (*Cont'd.*).

Corresponding Formula of Table I	Kinds of Resistances Referred to $(R_\gamma R_\delta)$	Coefficients Name	Coefficients Value	Index Field of Variation	Number of Coefficients	Second-Order Derivatives $\left(\frac{\partial^2 R_p}{\partial R_\gamma \partial R_\delta}\right)_0$
9	R_{al}, R	$\overset{(0)}{G}_{ali-(i+k)}$	$\frac{i}{10R}$	$1 \le i \le 7$; $2 \le i+k \le 10$; $k>0$	42	$-\frac{i(10-i)10^{-3}}{R}$
		$\overset{(0)}{G}_{ali-(i-k)}$	$-\frac{10-i}{10R}$	» ; $1 \le i-k \le 7$; $k \ge 0$	28	$\frac{(10-i)^2 10^{-3}}{R}$
10[(*)]	R_{vl}, R_{at}	$\overset{(0)}{G}_{vli-at(i+2k-1)}$	$\frac{i}{10R_{vt}}$	$3 \le i \le 5'$; $4 \le i+2k-1 \le 9$; $k>0$	8	$-\frac{2i^2 10^{-3}}{R_{vt}}$
		»	$-\frac{i}{10R_{vt}}$	$5'' \le i \le 7$; $6 \le i+2k-1 \le 9$; »	5	$\frac{2i(10-i)10^{-3}}{R_{vt}}$
		»	$\frac{i}{20R_{vt}}$	$i=3;5'$; $i+2k-1=10$	2	$-\frac{i^2 10^{-3}}{2R_{vt}}$
		»	$-\frac{i}{20R_{vt}}$	$i=5'';7$; »	2	$\frac{i(10-i)10^{-3}}{2R_{vt}}$
		$\overset{(0)}{G}_{vli-at(i-2k+1)}$	$-\frac{10-i}{10R_{vt}}$	$3 \le i \le 5'$; $1 \le i-2k+1 \le 4$; $k>0$	5	$\frac{2i(10-i)10^{-3}}{R_{vt}}$
		»	$\frac{10-i}{10R_{vt}}$	$5'' \le i \le 7$; $1 \le i-2k+1 \le 6$; »	8	$-\frac{2(10-i)^2 10^{-3}}{R_{vt}}$
		»	$-\frac{10-i}{20R_{vt}}$	$i=3;5'$; $i-2k+1=0$	2	$\frac{(10-i)i\,10^{-3}}{2R_{vt}}$
		»	$\frac{10-i}{20R_{vt}}$	$i=5'';7$; »	2	$-\frac{(10-i)^2 10^{-3}}{2R_{vt}}$
11[(*)]	R_{vl}, R	$\overset{(0)}{G}_{vli-i}$	$\frac{(10-i)R_{at}}{20R_{vt}R}$	$3 \le i \le 5'$	3	$-\frac{i(10-i)R_{at}10^{-2}}{20R_{vt}R}$
		»	$-\frac{(10-i)R_{at}}{20R_{vt}R}$	$5'' \le i \le 7$	»	$\frac{(10-i)^2 R_{at}10^{-2}}{20R_{vt}R}$
		$\overset{(0)}{G}_{vli-(i-k)}$	$\frac{(10-i)R_{at}}{20R_{vt}R}$	$3 \le i \le 5'$; $1 \le i-k \le 4$; $k>0$	9	$-\frac{i(10-i)R_{at}10^{-2}}{20R_{vt}R}$
		»	$-\frac{(10-i)R_{at}}{20R_{vt}R}$	$5'' \le i \le 7$; $1 \le i-k \le 6$; »	15	$\frac{(10-i)^2 R_{at}10^{-2}}{20R_{vt}R}$
		$\overset{(0)}{G}_{vli-(i+k)}$	$-\frac{i R_{at}}{20R_{vt}R}$	$3 \le i \le 5'$; $4 \le i+k \le 10$; »	18	$\frac{i^2 R_{at}10^{-2}}{20R_{vt}R}$
		»	$\frac{i R_{at}}{20R_{vt}R}$	$5'' \le i \le 7$; $6 \le i+k \le 10$; »	12	$-\frac{i(10-i)R_{at}10^{-2}}{20R_{vt}R}$
12	R_{at}, R_{vt}	$\overset{(0)}{G}_{ati-vti}$	$-\frac{4}{5R_{vt}}$	$1 \le i \le 9$	9	$\frac{32 \cdot 10^{-3}}{R_{vt}}$
		»	$-\frac{9}{20R_{vt}}$	$i=0;10$	2	$\frac{9 \cdot 10^{-3}}{2R_{vt}}$
		$\overset{(0)}{G}_{ati-vt(i\pm2k)}$	$\frac{1}{5R_{vt}}$	$1 \le i \le 9$; $1 \le i\pm2k \le 9$; $k>0$	32	$-\frac{8 \cdot 10^{-3}}{R_{vt}}$
		»	$\frac{1}{10R_{vt}}$	» ; $i\pm2k=0;10$	8	$-\frac{2 \cdot 10^{-3}}{R_{vt}}$
		»	»	$i=0;10$; $1 \le i\pm2k \le 9$; $k>0$	»	»
		»	$\frac{1}{20R_{vt}}$	» ; $i\pm2k=0;10$; $k=5$	2	$-\frac{10^{-3}}{2R_{vt}}$

*See footnote[a] of Table VII.

R_{vt}: The factor $1/R_{vt}$ appears in formulas 3, 4, 7, 10, and 11; that requires R_{vt} to be high. The factor $(\sigma_{R_{vt}}/R_{vt})$ is in formulas 5, 8, 12, and 14; this means that the magnitude of R_{vt} must not affect its precision (because of stray resistances, for instance). A compromise between the two exigencies should be aimed at.

R: Resistances R are the main resistances; then their sizes are usually conditioned by exigencies other than the accuracy of the result. At any rate, it might be interesting to remark which conditions are the best ones for the accuracy of results. It can be said once more that the best condition is the result of a compromise; the factor $1/R$ is in formula 2 (high R); the

TABLE VIII (*Cont'd.*).

Corresponding Formula of Table I	Kinds of Resistances Referred to $(R_\gamma R_\delta)$	Coefficients Name	Coefficients Value	Index Field of Variation	Number of Coefficients	Second-Order Derivatives $\left(\frac{\partial^2 R_p}{\partial R_\gamma \partial R_\delta}\right)_0$
13	R_{at}, R	$G^{(0)}_{ati-(i+k)}$	$\frac{4}{5R}$	$1 \le i \le 9;\ k=0;1$	18	$-\frac{16\cdot 10^{-3}}{R}$
		»	$\frac{9}{10R}$	$i=0;10;\ i+k=1;10;\ k=0;1$	2	$-\frac{9\cdot 10^{-3}}{R}$
		$G^{(0)}_{ati-(i-k)}$	$-\frac{1}{5R}$	$2 \le i \le 9;\ 1 \le i-k \le 8;\ k>0$	36	$\frac{4\cdot 10^{-3}}{R}$
		$G^{(0)}_{ati-(i+k+1)}$	»	$1 \le i \le 8;\ 3 \le i+k+1 \le 10;$ »	»	»
		$G^{(0)}_{at0-i}$	$-\frac{1}{10R}$	$2 \le i \le 10$	9	$\frac{10^{-3}}{R}$
		$G^{(0)}_{at10-(11-i)}$	»	»	»	»
14	R_{vt}, R	$G^{(0)}_{vti-(i+k)}$	$\frac{4R_{at}}{5R_{vt}R}$	$1 \le i \le 9;\ k=0;1$	18	$-\frac{16R_{at}10^{-3}}{R_{vt}R}$
		»	$\frac{9R_{at}}{10R_{vt}R}$	$i=0;10;\ i+k=1;10;\ k=0;1$	2	$-\frac{9R_{at}10^{-3}}{R_{vt}R}$
		$G^{(0)}_{vti-(i-k)}$	$-\frac{R_{at}}{5R_{vt}R}$	$2 \le i \le 9;\ 1 \le i-k \le 8;\ k>0$	36	$\frac{4R_{at}10^{-3}}{R_{vt}R}$
		$G^{(0)}_{vti-(i+k+1)}$	»	$1 \le i \le 8;\ 3 \le i+k+1 \le 10;$ »	»	»
		$G^{(0)}_{vt0-i}$	$-\frac{R_{at}}{10R_{vt}R}$	$2 \le i \le 10$	9	$\frac{R_{at}10^{-3}}{R_{vt}R}$
		$G^{(0)}_{vt10-(11-i)}$	»	»	»	»
15	R	$G^{(0)}_{i-i}$	$\frac{9}{10R}$	$1 \le i \le 10$	10	$-\frac{18\cdot 10^{-3}}{R}$
	R, R	$G^{(0)}_{i-k}$	$-\frac{1}{10R}$	» $;1 \le k \le 10;\ k \ne i$	45	$\frac{2\cdot 10^{-3}}{R}$

factor (σ_R/R) is in formulas 6, 9, 11, 13, and 14 (high precision); the factor $(\sigma_R/R)\ \sigma_R$ is in formula 15 (low values of σ_R/R and of σ_R).

VII. Results of Computations Carried Out on a Particular Buildup Box

Measurements have been performed for determining the values of all the resistances that appear in the equivalent circuit of Fig. 1. The values that are necessary for the computations are reported in Table IV. Notice that the measurement accuracy corresponds to the significant figures of the values; exception must be made for the R_{ati}, where the contact resistances are very important. The values that have been reported represent the results of a set of measurements; many other sets have been determined for obtaining the mean and the standard deviation of R_{ati}.

The results of computations are reported in Tables V and VI. Table V contains the variations computed by means of the formulas of Table I. Table VI contains these variations (repeated for allowing an easier comparison) and the means and standard deviations computed by means of the formulas of Table III; by comparing the variations and the standard deviations it is possible to conclude that the computations of standard deviations are certainly sufficient for obtaining the magnitude of the uncertainties.

Appendix

A. Calculation of The Coefficients of Cohn's Theorem

Coefficients $\beta_{\gamma a}$ and $\beta_{\gamma v}$ are easily calculated, without introducing approximations, from their definition, explained in Fig. 2(a) and (b), by considering the current distribution when port a or port v are supplied. By applying symmetry considerations developed in Section III-B, on the network of Fig. 1 (after imposing Assumptions 1 and 2 in order to get the base circuit), the values of Table VII are obtained[7] (in this table the corresponding first derivatives are reported). It is interesting to notice that these coefficients do not depend on the values of the resistances that appear in the base circuit; ob-

[7]For understanding the meanings of notations an example is useful: coefficient β_{ali-a} is the ratio between the current in resistance R_{ali} and the current supplied by port a. Moreover, notice that the assumed directions for the various currents in the network branches of Fig. 1 are used to give the algebraic sign both to coefficients β and to coefficients $G^{(0)}$.

viously, this is a consequence of the particular symmetry of the network.

Calculations of conductances $G^{(\circ)}$ are made according to the definition explained in Fig. 2(c), assuming approximation hypotheses mentioned in Section III-C. The values of conductances $G^{(\circ)}$ and of the corresponding second derivatives are reported in Table VIII.[8]

B. Comparisons With The Results of Other Authors

Riley's Formulas: It is necessary to remark that Riley has calculated the values of errors by a perturbation technique in the simple case of two resistors connected in parallel, then has obtained a sort of probable error in the case of the parallel connection of m main resistors by extrapolation. The easy calculation of first- and second-order effects of resistances R makes exception; they are calculated directly on the network of m main resistances and coincide perfectly with those expressed by formula 15 of Table I.

The formula that gives the effects of R_g, calculated on the simplified circuit, is affected, in my opinion, by a trivial mistake: instead of a factor of $\frac{1}{2}$ there should be a factor of $\frac{1}{4}$; with such a correction the formula coincides with the correspondent one that may be easily derived on the same simplified circuit by using Cohn's theorem. As for the formula that gives the probable error for the complete buildup box by a comparison of sizes (see also formula 1 of Table III), the result is too pessimistic.

Riley develops calculations also for the combined effects of R_{at} and R_{vt} and of R_{at} and R. The formulas derived from calculations on the simplified circuit substantially agree with the formulas attained by application of Cohn's theorem on the same circuit. As for the formulas pertaining to the complete resistance box, they are in good agreement, as for the sizes and the structures, with formulas 12 and 13 of Tables I and III.

Page's Formulas: First, let some differences be pointed out between Page's way and ours. No assumption is made by Page on the sizes of resistances R_{at}, R_{vt}, and R, while in this paper it is assumed that $R_{at} \ll R_{vt}$ (see[6]) and $R_{at} \ll R$; nevertheless approximations in Page's calculation follow from expanding in series some inverse matrices. These two different ways should correspond to the same approximations of results; as a matter of fact differences of results remain, which could, or could not, be due to the different approximations.

Page has calculated, as a main error term, a formula for the effects of R_{at} and R_{vt}, which can be written

$$(\Delta R_p)_{R_{at}-R_{vt}} = \frac{4 \cdot 10^{-2}}{R_{at} + R_{vt}} \left(\sum_{i=0}^{10} \Delta R_{ati}\, \Delta R_{vti} + \Delta R_{ato}\, \Delta R_{vto} + \Delta R_{at10}\, \Delta R_{vt10} \right). \quad (4)$$

If it is assumed that $R_{at} \ll R_{vt}$, (4) coincides with the first term of formula 12 of Table I; the term dealing with the couples of resistances R_{ati}, R_{vtj} $(i \neq j)$ is missing.

As secondary terms, Page has also calculated the effects (which are ordinarily negligible, indeed) of the couples of resistances R_{at}, R and R_{vt}, R. His formula for the effects of R_{at} and R coincides exactly with formula 13 of Table I, if it is assumed that $R_{at} + R_{vt} = R_{vt}$; under the same assumption Page's formula for the effects of R_{vt} and R and formula 14 of Table I coincide too.

Finally, Page's formula for the second-order effects of resistances R is

$$(\Delta R_p)_R = \frac{10^{-2} \cdot R_{at} \cdot R_{vt}}{(R_{at} + R_{vt}) R^2} \left[2 \sum_{i=1}^{10} (\Delta R_i)^2 + 2 \sum_{i=1}^{9} \Delta R_1 \cdot \Delta R_{(i+1)} + (\Delta R_1)^2 + (\Delta R_{10})^2 \right]. \quad (5)$$

This formula is quite different from formula 15 of Table I (which coincides, as noted, with Riley's corresponding formula).

The presence of factors of R_{at} and R_{vt} is quite odd; it should mean that the second-order effects of resistances R become zero when either R_{at} or R_{vt} are zero. It is difficult to explain such a behavior that contradicts both the results of this paper (Riley's) and intuition.

Acknowledgment

The author would like to thank Prof. R. Marenesi and Prof. R. Sartori for their useful suggestions given during the entire work, and E. Arri, Chief of the Department of I.E.N., where the measurements were carried out.

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[8] The meanings of notations are explained by an example. Conductance $G^{(\circ)}_{vli-at(i+2k-1)}$ is the mutual conductance between resistance R_{vli} and resistance $R_{at(i+2k-1)}$.