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A little-known minimum concerning resistors in series and in parallel

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Abstract

In the elementary theory of electrical circuits, the series connection of n identical resistors has an equivalent resistance n^2 greater than the equivalent resistance of their parallel connection. More briefly, the series/parallel ratio is n^2. We show that if the resistors are not all identical, the series/parallel ratio is greater than n^2 and can never be less than n^2. This little-known minimum has been demonstrated previously, using the theorem that the arithmetic mean of non-negative numbers always equals or exceeds their geometric mean. Here we present a simple proof that avoids using the theorem. If the n resistors differ only slightly, the series/parallel ratio is still n^2 to first order in their differences. Because this 'weaker' form has long been known, we discuss only briefly its significance for electrical metrology. We present a Monte Carlo simulation of the series/parallel ratio for two resistors, one of which varies in accordance with a Gaussian density distribution defined by three tolerance ranges, and we compare the simulation graphically with the theoretical series/parallel ratio for each of these ranges. This theoretical ratio is essentially a plot of the density distribution of the square of a Gaussian variable, or equivalently of a chi-squared distribution on one degree of freedom. Finally, we note an interesting connection between the minimum and the second law of thermodynamics.

Keywords: resistor, series, parallel, asymmetry, voltage divider, tolerance, Monte Carlo

1. Introduction

The connection of resistors in series and in parallel is fundamental to the study of electrical circuits [1]. The equivalent resistance R_{series} of n resistors R_1, R_2,...,R_n in series can be shown to be given by Eq. (1), and the equivalent resistance R_{parallel} of those resistors in parallel can be shown to be given by Eq. (2).

$$R_{\text{series}} = R_1 + R_2 + ... + R_n. \tag{1}$$

$$\frac{1}{R_{\text{parallel}}} = \frac{1}{R_1} + \frac{1}{R_2} + ... + \frac{1}{R_n}. \tag{2}$$

If the n resistors are all identical, it immediately follows from Eqs. (1) and (2) that the series/parallel ratio $R_{\text{series}}/R_{\text{parallel}} = n^2$. The more general case when the resistors are not identical shows an interesting and unexpected minimum as explained in Section 2 and also in Appendix A.

2. The little-known minimum

Eqs. (1) and (2) give

$$\frac{R_{\text{series}}}{R_{\text{parallel}}} = (R_1 + R_2 + ... + R_n)\left(\frac{1}{R_1} + \frac{1}{R_2} + ... + \frac{1}{R_n}\right). \tag{3}$$

Multiplying out the brackets in Eq. (3) gives us:

$$\frac{R_{\text{series}}}{R_{\text{parallel}}} = n + \left(\frac{R_1}{R_2} + \frac{R_2}{R_1}\right) + \left(\frac{R_1}{R_3} + \frac{R_3}{R_1}\right) + ... + \left(\frac{R_2}{R_3} + \frac{R_3}{R_2}\right) + ... \tag{4}$$

$$= n + \sum_{i=1}^{n} \sum_{j>i}^{n} \left(\frac{R_i}{R_j} + \frac{R_j}{R_i} \right). \tag{5}$$

Each term in brackets on the right side of Eqs. (4) or (5) is of the form of $z = \left(\frac{x}{y} + \frac{y}{x}\right)$ as considered in Appendix A, where it is shown that for x, y non-negative, z can never be less than 2. Such paired mutual reciprocals have been observed to occur in general circuits involving combinations of equal resistors and their resulting equivalent resistance [2]. Defining a_{ij} as the excess of $\left(\frac{R_i}{R_j} + \frac{R_j}{R_i}\right)$ over 2,

$$a_{ij} = \left(\frac{R_i}{R_j} + \frac{R_j}{R_i} \right) - 2, \tag{6}$$

where a_{ij} is non-negative for all $i, j,\ j > i$, we may now write Eq. (5) as

$$\frac{R_{\text{series}}}{R_{\text{parallel}}} = n + \sum_{i=1}^{n} \sum_{j>i}^{n} (2 + a_{ij}), \tag{7}$$

where all the a_{ij} are non-negative. With $j > i$ there are $\frac{1}{2}n(n-1)$ terms being summed in the summation term on the right side of Eq. (7). For example, if $n = 4$ there are $\frac{1}{2} \times 4 \times 3 = 6$ terms being summed, these being $2 + a_{12}$, $2 + a_{13}$, $2 + a_{14}$, $2 + a_{23}$, $2 + a_{24}$, $2 + a_{34}$. The sum of the six 2's, in this example, is 12. In general, Eq. (7) therefore gives

$$\frac{R_{\text{series}}}{R_{\text{parallel}}} = n + \left(2 \times \frac{1}{2} n(n-1) \right) + \sum_{i=1}^{n} \sum_{j>i}^{n} a_{ij} = n + n(n-1) + \sum_{i=1}^{n} \sum_{j>i}^{n} a_{ij} = n^2 + \sum_{i=1}^{n} \sum_{j>i}^{n} a_{ij}. \tag{8}$$

Our result is therefore that the ratio of the effective resistance of n resistors in series to the effective resistance of the same resistors in parallel can never be less than n^2. This n^2 is the minimum in the title of this paper and represents an unexpected asymmetry. By contrast, symmetry would describe the case where, depending on the choice of resistors, the series/parallel ratio would be as likely to be less than n^2 as to exceed n^2.

To take a slightly different approach leading to Eq. (7), and verifying (as remarked after Eq. (7)) that there are $\frac{1}{2}n(n-1)$ terms being summed, we note that in Eq. (3) there are n summed terms in each of the two brackets being multiplied, so that their product has n^2 summed terms. Of these n^2 terms, n take the value 1 (for example, $R_1 \times \frac{1}{R_1} = 1$) and therefore sum to n, which is accordingly the first term on the right-hand side of Eq. (4). This accounts for n of the n^2 terms and so there must remain $n^2 - n$ terms of the form $\frac{R_i}{R_j}$, or $\frac{1}{2}(n^2 - n) = \frac{1}{2}n(n-1)$ terms where $\frac{R_i}{R_j}$ is paired with its reciprocal $\frac{R_j}{R_i}$ as in the rest of the right-hand side of Eq. (4).

We refer to the asymmetry as the 'strong' version of the result for a series/parallel ratio. When all the resistors differ from one another only slightly, we shall refer to the result as the 'weak' version.

There exists an extensive literature on resistor networks: simpler aspects of circuit theory [2-5], more complicated resistor networks, often involving identical resistors and with applications such as the generation of irrational numbers [6-14], analogies with other areas of physics [4, 15], and consideration of the weak version and its application to electrical metrology [16-24]. We draw particular attention to Tykodi [15], where the strong version is derived. This derivation, however, makes use of the theorem that the arithmetic mean of any set of non-negative numbers must equal or exceed their geometric mean. The theorem involves some rather complicated mathematics, which is unnecessary as our simple proof demonstrates.

The strong version, and its associated asymmetry, are illustrated in Fig. 1, which is a histogram of the series/parallel ratio for $n = 3$ resistors, each of which independently takes the ten values 1 Ω, 2 Ω,...,10 Ω. The notation $9 < 10$ (for example) on the horizontal axis of Fig. 1 indicates that all the values in the associated column lie in the range 9.00 to 9.99. The histogram displays the 1000 values of the series/parallel ratio,which peaks at the value $n^2 = 9$ and then drops

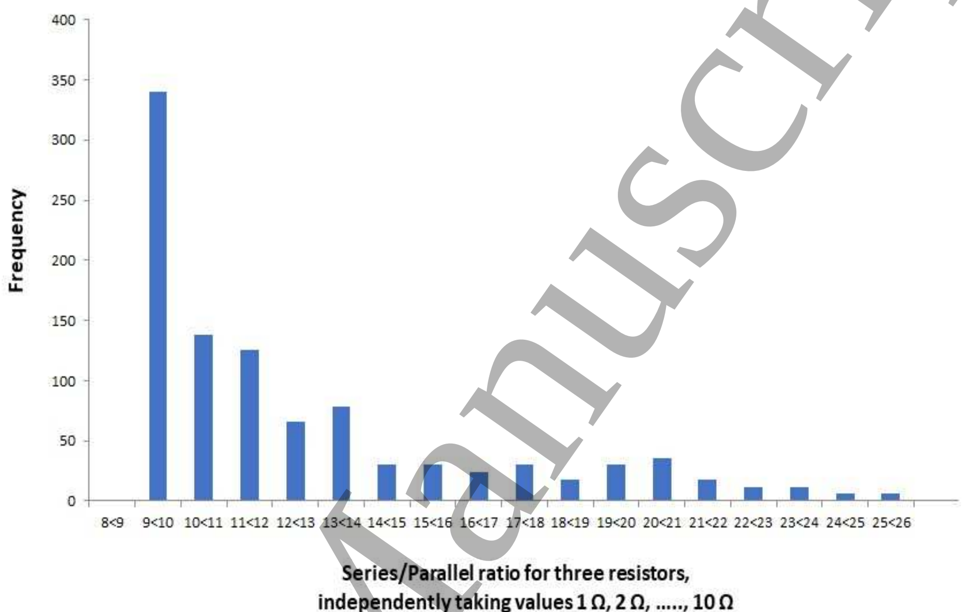

Figure 1: Fig. 1 Histogram of Series/Parallel ratio for three resistors. Each independently takes values 1 Ω, 2 Ω,...,10 Ω.

off but only on the positive side of 9. Values less than 9 do not exist.

3. The case where all resistors are approximately equal

The ratio $R_{\text{series}}/R_{\text{parallel}}$ when all resistors are approximately equal is n^2 and this is accurate to second-order in their deviations from one another. This result, which has long been known in the venerable discipline of electrical measurements [19,20] and has metrological applications [16-24], may be shown as follows. We write $R_i = R_0(1+\epsilon_i)$, $(i = 1, 2, ..., n)$, where R_0 is the common nominal value and ϵ_i is a small proportional deviation of R_i from R_0. Eq. (4) then gives

$$\frac{R_{\text{series}}}{R_{\text{parallel}}} = n+\left(\frac{1+\epsilon_1}{1+\epsilon_2}+\frac{1+\epsilon_2}{1+\epsilon_1}\right)+\left(\frac{1+\epsilon_1}{1+\epsilon_3}+\frac{1+\epsilon_3}{1+\epsilon_1}\right)+...+\left(\frac{1+\epsilon_2}{1+\epsilon_3}+\frac{1+\epsilon_3}{1+\epsilon_2}\right)+...+ \tag{9}$$

We use the following approximation up to second-order: $\frac{1}{1+\epsilon_i} \approx (1-\epsilon_i+\epsilon_i^2)$, for $i = 1, 2, ..., n$. It is then, incidentally, simple to show that in Eq. (6), $a_{ij} = (\epsilon_i - \epsilon_j)^2$. Eq. (9) gives:

$$\begin{aligned}\frac{R_{\text{series}}}{R_{\text{parallel}}} &= n + \left[(1+\epsilon_1)(1-\epsilon_2+\epsilon_2^2)+(1+\epsilon_2)(1-\epsilon_1+\epsilon_1^2)\right]+\\ &\left[(1+\epsilon_1)(1-\epsilon_3+\epsilon_3^2)+(1+\epsilon_3)(1-\epsilon_1+\epsilon_1^2)\right]+...+\\ &\left[(1+\epsilon_2)(1-\epsilon_3+\epsilon_3^2)+(1+\epsilon_3)(1-\epsilon_2+\epsilon_2^2)\right]+...\end{aligned} \tag{10}$$

In each pair of square brackets we have the form

$$(1+\epsilon_i)(1-\epsilon_j+\epsilon_j^2)+(1+\epsilon_j)(1-\epsilon_i+\epsilon_i^2)$$

and keeping terms up to and including second-order this is

$$(1-\epsilon_j+\epsilon_j^2+\epsilon_i-\epsilon_i\epsilon_j)+(1-\epsilon_i+\epsilon_i^2+\epsilon_j-\epsilon_i\epsilon_j)$$

$$= 2+(\epsilon_i^2+\epsilon_j^2-2\epsilon_i\epsilon_j) = 2+(\epsilon_i-\epsilon_j)^2.$$

As stated previously, there are $\frac{1}{2}n(n-1)$ such terms, and so we have

$$\frac{R_{\text{series}}}{R_{\text{parallel}}} = n + \left(2 \times \frac{1}{2}n(n-1)\right) + \sum_{i=1}^{n}\sum_{j>i}^{n}(\epsilon_i - \epsilon_j)^2 \tag{11}$$

$$= n^2 + \text{terms of order } \epsilon^2. \tag{12}$$

We note that the second-order correction in Eq. (12) must be positive, and it is straightforward to show (see Appendix B) that Eq. (11) may be written

$$\frac{R_{\text{series}}}{R_{\text{parallel}}} = n^2(1 + \text{var } \epsilon), \tag{13}$$

where var ϵ is the variance of the ϵ, defined in the standard form as

$$\text{var } \epsilon = \frac{1}{n}\sum_{i=1}^{n}(\epsilon_i - \bar{\epsilon})^2 \tag{14}$$

with $\bar{\epsilon} = \frac{1}{n}\sum_{i=1}^{n}\epsilon_i$ as the mean of the ϵ. As shown in Section 2, however, the series/parallel ratio must equal or exceed n^2 not only for nominally equal resistors, but also for any set of n resistor values.

4. A simple laboratory demonstration of the series/parallel results

An experiment, suitable for an undergraduate laboratory and which demonstrates the weak version, could be based on the following set-up. In Fig. 2, a voltage divider is shown in which the stabilised input voltage V_{in} produces an output voltage V_{out} one-fifth of the input voltage. The divider comprises two high-accuracy 100 Ω resistors in series, and two high-accuracy 100 Ω resistors in parallel. This is therefore the case $n = 2$, which implies a voltage division by $n^2+1 = 5$. Fig. 3 shows a voltage divider in which one of the series resistors, and one of the parallel resistors, have values 10% larger than 100 Ω, and this extra 10 Ω resistance is depicted explicitly as an additional resistor in series. It is easily shown that the output voltage V_{out} is now 0.199637 of the input voltage. So a large mismatch of 10% in two of the resistors, respectively in the series arm and the parallel arm, results in a voltage division only 0.18% different from the desired value of 1/5. The series-parallel connection of Fig. 3 therefore 'attenuates' the mismatch of 10% by a factor of $10/0.18 \approx 56$. Students could be asked to determine experimentally the discrepancy from the desired value of 1/5 for larger mismatches of, say, 20% (an extra 20 Ω) and 30% (30 Ω). The theoretical values of discrepancy are respectively 0.66% (attenuation by ≈ 30) and 1.37% (attenuation by ≈ 22).

This insensitivity of a series/parallel voltage divider ratio to a mismatch in the dividing resistors underpins the design of the high-precision voltage dividers required in electrical metrology [22]. In [23] the case $n = 3$ is discussed, where the input voltage is divided by $n^2 + 1 = 10$. We do not discuss the further refinements to the design of high-precision voltage dividers [23], since this area of electrical metrology is based on the widely-known weak version.

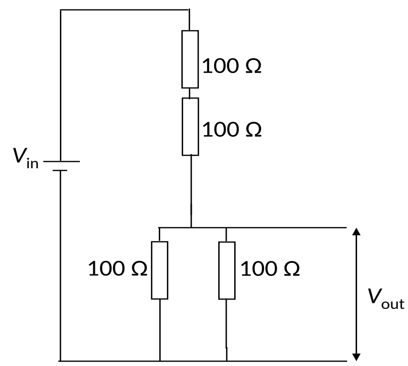

Figure 2: Fig. 2 Voltage divider with high-accuracy resistors: two 100 Ω resistors in series, and two 100 Ω resistors in parallel. Output 1/5 of input

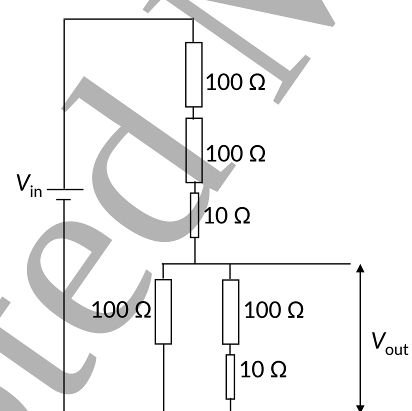

Fig. 3 Voltage divider with one series and one parallel resistor 10% larger than nominal

5. Monte Carlo simulation of the series/parallel ratio for two resistors, nominally equal but one of which varies slightly

In this simulation, the ratio $R_{\text{series}}/R_{\text{parallel}}$ is calculated for two resistors ($n = 2$). One resistor is fixed at 100 Ω. The other has a nominal resistance of 100 Ω, but varies slightly from 100 Ω. Values of $(R_{\text{series}}/R_{\text{parallel}}) - 4$ were calculated for 10000 trials at each of three tolerance levels: 2%, 5% and 10%. These percentage figures represent three standard deviations of a Gaussian density distribution, such that, for example with 2% tolerance, 99.7% of values are expected to be within 2% of the nominal value with mean value 100 Ω. As predicted, $R_{\text{series}}/R_{\text{parallel}}$ is never less than $n^2 = 4$, so that $(R_{\text{series}}/R_{\text{parallel}}) - 4$ is always positive.

If the fixed resistance is denoted R and the variable resistance is denoted $R(1+\epsilon)$, where ϵ is a small quantity distributed as a Gaussian with mean zero and standard deviation σ, then for 2% tolerance we have $\sigma = 0.02/3 = 0.006667$, for 5% tolerance $\sigma = 0.05/3 = 0.01667$ and for 10% tolerance $\sigma = 0.1/3 = 0.03333$. Gaussian density distributions, which in principle must yield negative values, are often assumed to describe quantities that, as in this case, cannot be negative; however, the probability of negative values in such cases is vanishingly small. It is straightforward to show that $R_{\text{series}}/R_{\text{parallel}} = 4 + \epsilon^2$ to second order. The density distribution of $(R_{\text{series}}/R_{\text{parallel}}) - 4$ therefore is equivalent to that of the square of a Gaussian variable. This is otherwise known as a chi-squared density distribution on 1 degree of freedom. Calling $y = \epsilon^2$, the density distribution, or equivalently the probability density function, $f(y)$ of y is

$$f(y) = \frac{1}{\sigma\sqrt{2\pi}} y^{-1/2} \exp\left(-y/(2\sigma^2)\right) \tag{15}$$

which increases to infinity at $y = 0$. However, the total area enclosed by $f(y)$ is finite at the value 1, and the mean value $\bar{y}$ of y is $\bar{y} = \sigma^2$ and the standard deviation

$s(y)$ of y is $s(y) = \sigma^2\sqrt{2}$.

The results of the 10000 trials are shown in Fig. 4, for each of the three tolerance levels. The notation along the horizontal axis is the same as was described earlier in connection with Fig. 1. Fig. 5 shows the corresponding theoretical graphs plotted following Eq. (15).

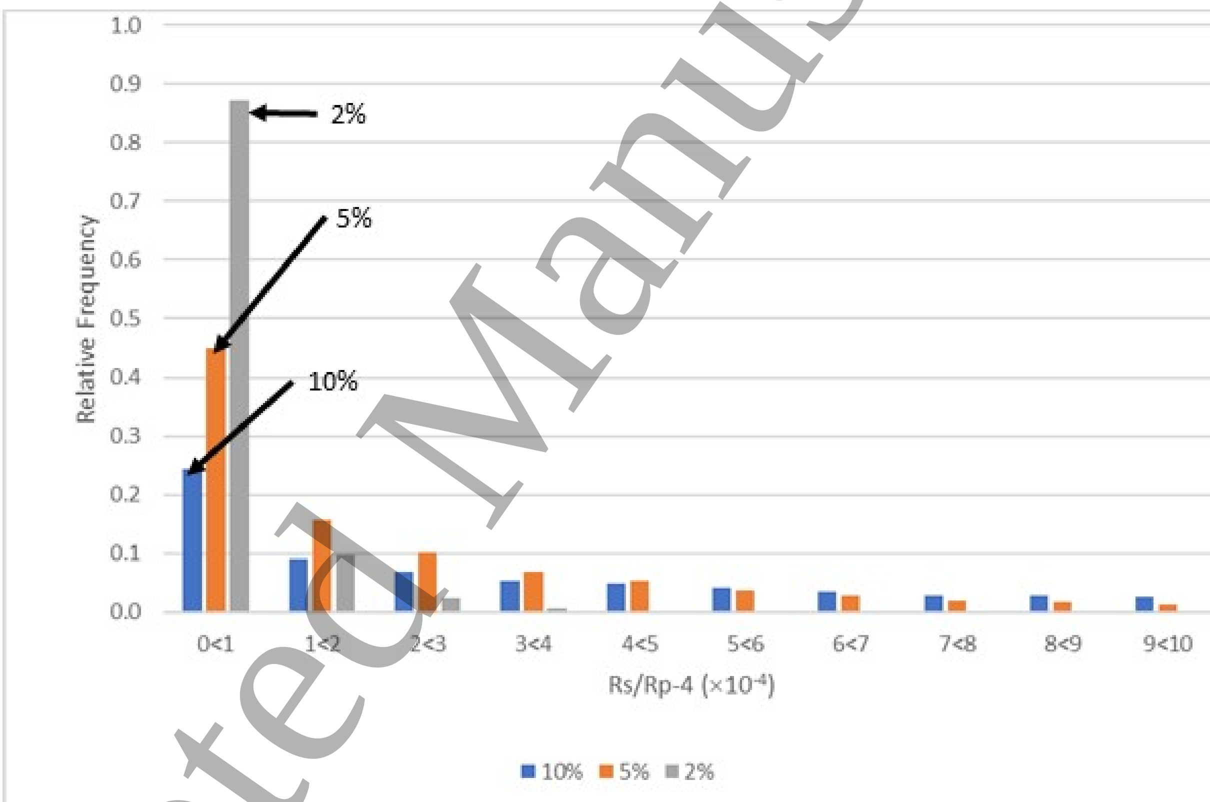

Fig. 4 Histogram of Monte Carlo simulation for two resistors, showing dependence of the increase in series/parallel ratio R_s/R_p over 4 with resistor tolerance as parameter

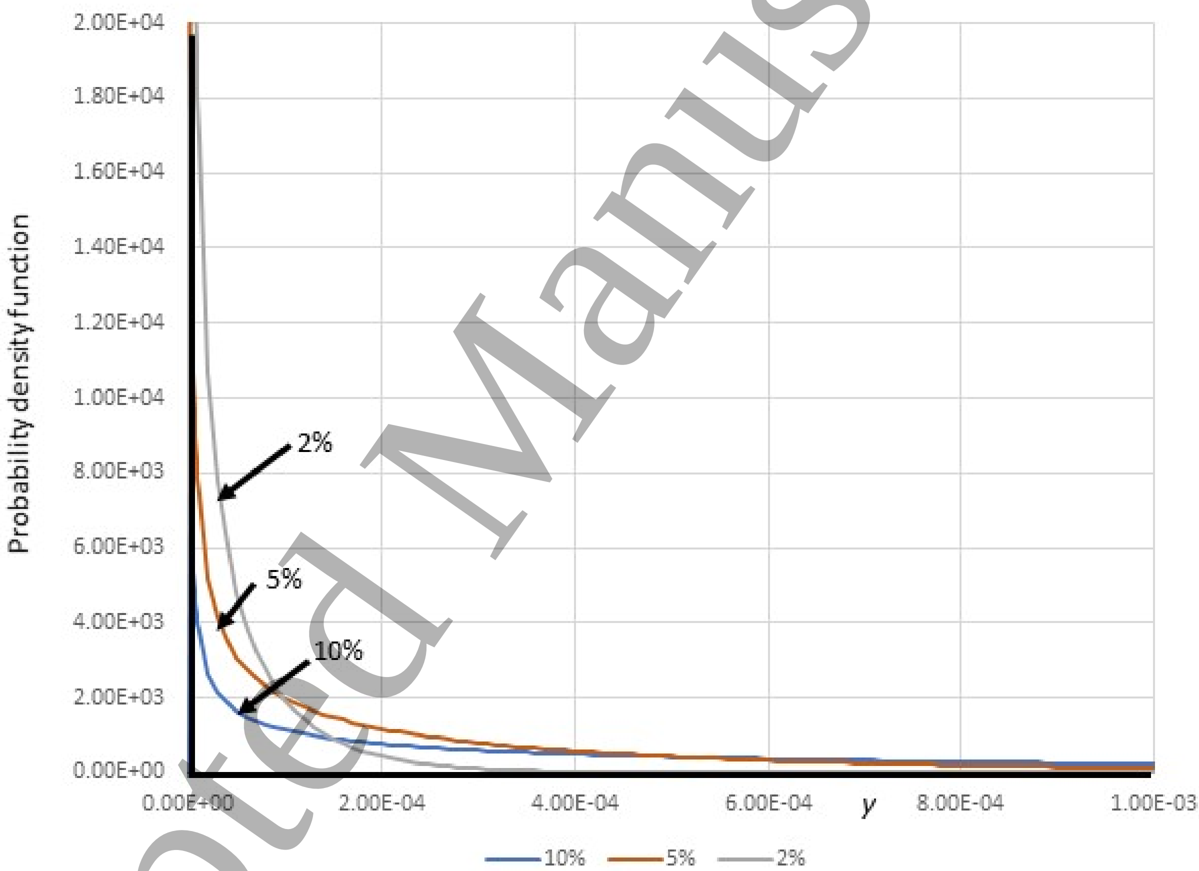

Fig. 5 Theoretical plots of chi-square density distribution on 1 d.o.f., corresponding to Fig. 4

6. Conclusion

The ratio of the effective resistance of n resistors in series, to the effective resistance of the same n resistors in parallel, cannot be less than n^2. From our search of the literature, this asymmetry appears to be little-known. A weaker version of the same result has long been known: if the resistors are approximately equal but with small deviations from equality, the ratio departs from n^2 only by deviations of the second order. This weaker version has found use in electrical metrology as described in section 3. The departure from n^2 is always in the direction of a very slight increase when the resistors are approximately equal, and this is consistent with the strong version. Through carrying out the experiment described in Section 4 and the Monte Carlo simulation where students are given scope to vary the number and values of resistors and their tolerances, students may be expected to gain a deeper insight into circuit theory. The Monte Carlo simulation represents an excellent opportunity to introduce this widely used and powerful technique to undergraduate students [26].

Appendix A. Range of values of a mathematical form involving non-negative numbers

Consider the form

$$z = \left(\frac{x}{y} + \frac{y}{x}\right) \tag{16}$$

where x and y are non-negative real numbers. When $x = y$, then of course $z = 2$. We show that z can never be less than 2. Writing

$$z = \left(\frac{x^2 + y^2}{xy}\right), \tag{17}$$

we equate the right side of Eq. (17) to $2 + a$, where a is any real number:

$$\left(\frac{x^2 + y^2}{xy}\right) = 2 + a, \tag{18}$$

so that

$$x^2 + y^2 = 2xy + axy. \tag{19}$$

This may be written

$$(x - y)^2 = axy. \tag{20}$$

The left side of Eq. (20) must be non-negative. Hence the right side of Eq. (20) is also non-negative, and since x and y are non-negative, so is a. The left side of Eq. (18), and equivalently the right side of Eq. (16), cannot therefore be less than 2. This is a counter-intuitive result, since when $x = y$, then $z = 2$, but although x and y occur symmetrically in Eq. (16) (in the sense that interchanging x and y leaves z unchanged), z cannot be less than 2.

Appendix B. The series/parallel ratio in terms of the variance of approximately equal resistors

The statistical term 'variance' var x of a sample of n measurements x_1, x_2, ...x_n is defined as

$$\text{var } x = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2 \tag{21}$$

where $\bar{x}$ is the mean value of the n measurements,

$$\bar{x} = \frac{1}{n}(x_1 + x_2 + ... + x_n). \tag{22}$$

The positive square root of the variance is a measure of the spread of the measurements, and is generally called the standard deviation, as in Section 5. Both the variance and standard deviation are 'origin-independent'; that is, if the same constant is added to or subtracted from each of the n measurements, the variance and standard deviation are unaffected. To simplify this analysis, we may therefore define n terms x'_1, x'_2,...x'_n as

$$x'_i = x_i - \bar{x} \tag{23}$$

for $i = 1, 2, ..., n$. We now have

$$\text{var } x' = \text{var } x = \frac{1}{n}\sum_{i=1}^{n} x_i'^2 \tag{24}$$

in view of the origin-independence of the variance. We also have

$$\sum_{i=1}^{n} x_i' = 0, \tag{25}$$

and therefore

$$\left(\sum_{i=1}^{n} x_i'\right)^2 = 0 \tag{26}$$

implying that

$$x_1'^2 + x_2'^2 + ... + x_n'^2 + 2P = 0 \tag{27}$$

where P is the sum of all pairwise products

$$P = x_1'x_2' + x_1'x_3' + ... + x_2'x_3' + x_2'x_4' + ... + x_{n-1}'x_n'. \tag{28}$$

In Eq. (11), the last term on the right-hand side is

$$\sum_{i=1}^{n}\sum_{j>i}^{n}(\epsilon_i - \epsilon_j)^2$$

or, calling this term F, and in the more general notation used here

$$F = \sum_{i=1}^{n}\sum_{j>i}^{n}(x_i' - x_j')^2 \tag{29}$$

since (from Eq. (23)) $x_i' - x_j' = x_i - x_j$ for all $i, j = 1, 2, ..., n$. Writing Eq. (29) out in full

$$F = (x_1' - x_2')^2 + (x_1' - x_3')^2 + +... + (x_2' - x_3')^2 + ... + (x_{n-1}' - x_n')^2. \tag{30}$$

Each of the squared terms $x_1'^2$, $x_2'^2$,...,$x_n'^2$ appears $n-1$ times in Eq. (30), so that

$$F = (n-1)(x_1'^2 + x_2'^2 + ... + x_n'^2) - 2P. \tag{31}$$

But we have seen (Eq. (27)) that

$$x_1'^2 + x_2'^2 + ... + x_n'^2 + 2P = 0. \tag{32}$$

Adding Eqs. (31) and (32),

$$F + 0 = F = n(x_1'^2 + x_2'^2 + ... + x_n'^2), \tag{33}$$

and so, using Eq. (24),

$$F = \sum_{i=1}^{n} \sum_{j>i}^{n} (x_i' - x_j')^2 = n^2 \mathrm{var}\ x' = n^2 \mathrm{var}\ x, \tag{34}$$

which verifies Eq. (13).

Appendix C. A mathematical connection with the second law of thermodynamics

We note an interesting consequence of the mathematically exact statement that the quantity z defined as the simple function in Eq. (16) cannot be less than 2. Consider two equal masses of water where the temperature of one mass is T_1 and the temperature of the other is T_2, with $T_2 \neq T_1$. When mixed adiabatically and at constant pressure, the mixture has a final temperature $\frac{1}{2}(T_1 + T_2)$. If C_p is the specific heat at constant pressure, the change ΔS of the entropy of the universe is [25]:

$$\Delta S = C_p \log\left(\frac{(T_1 + T_2)/2}{T_1}\right) + C_p \log\left(\frac{(T_1 + T_2)/2}{T_2}\right)$$

$$= C_p \log\left(\frac{(T_1+T_2)^2}{4T_1T_2}\right) = C_p \log\left(\frac{1}{4}\left[\frac{T_1}{T_2}+\frac{T_2}{T_1}\right]+\frac{1}{2}\right). \quad (35)$$

The quantity in square brackets in Eq. (35) is of the form of z in Eq. (16) and so cannot be less than 2. With $T_1 \neq T_2$, we therefore see that the change in entropy ΔS of the universe must be positive. This increase in entropy of the universe during a natural process, in our example the cooling of hot water and the warming of cold water when mixed, is one of several formulations of the second law of thermodynamics. Interestingly, the law is in essence statistical but is a consequence of the mathematically exact statement noted above.

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