Online Review Course of
Undergraduate Probability and Statistics

Review Lecture 6
Probability, part 3

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Course Website: www.lithoguru.com/scientist/statistics/review.html

Conditional Probability

- We judge probabilities based on what we know
  - What is the probability that Random Person X will develop lung cancer?
  - What is the probability that Random Person X will develop lung cancer given that person smokes?
- Let A and B be two events. We define the condition probability of A given B as
  - ℙ(A|B) = probability of A given that B has occurred

Conditional Probability

- ℙ(A|B) = probability of A given B
  - The A|B event occurs only when both A and B occur. Thus, ℙ(A|B) α ℙ(A∩B).
  - Since we know B has occurred, our new "universe" (effectively our new sample space) will be B
  - We can normalize our probabilities for the new "universe" by dividing by ℙ(B), since we want ℙ(A|B) to obey the axioms of probability

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

Conditional Probability Example

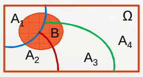
- I have two children, one of which is a girl. What is the probability that my other child is a girl?
  - Intuition says 0.5, but your intuition is wrong!
  - Let B = at least one girl, A = two girls
  - ℙ(A) = 0.25, ℙ(B) = 1- ℙ(B^c) = 0.75
  - We want ℙ(A|B)

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{\mathbb{P}(A)}{\mathbb{P}(B)} = \frac{1}{3}$$

Total Probability: Divide and Conquer

- Let $A_1 \ldots A_n$ be a partition of the sample space
- Given any event B, these events $A_1 \ldots A_n$ will also partition B

$$B = (A_1 \cap B) \cup (A_2 \cap B) \cup \cdots (A_n \cap B)$$



- By the additivity axiom,

$$\mathbb{P}(B) = \mathbb{P}(A_1 \cap B) + \mathbb{P}(A_2 \cap B) + \cdots + \mathbb{P}(A_n \cap B) = \sum \mathbb{P}(A_i \cap B)$$

- But, $\mathbb{P}(A_i \cap B) = \mathbb{P}(B|A_i)\mathbb{P}(A_i)$
- So,

$$\mathbb{P}(B) = \sum \mathbb{P}(B|A_i)\mathbb{P}(A_i)$$

(weighted average of conditional probabilities)

Total Probability: Common Partition

- One simple partition is A and A^c
- Applying the total probability formula to this partition,

$$\mathbb{P}(B) = \mathbb{P}(B|A)\mathbb{P}(A) + \mathbb{P}(B|A^c)\mathbb{P}(A^c)$$

- This will come in handy frequently

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Bayes' Theorem (Rule)

- From the definition of conditional probability we can write ℙ(A∩B) two ways

$$\mathbb{P}(A \cap B) = \mathbb{P}(A|B)\mathbb{P}(B)$$

$$\mathbb{P}(A \cap B) = \mathbb{P}(B|A)\mathbb{P}(A)$$

- Combining,

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(B|A)\mathbb{P}(A)}{\mathbb{P}(B)}$$ Bayes' Rule

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Bayes' Theorem Example

- Consider a lab test for a disease
  - It is 95% effective at detecting the disease
  - It has a false positive rate of 1%
  - The rate of occurrence of the disease in the general population is 0.5%
- I take the screening test and get a positive result. What is the likelihood I have the disease?

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Bayes' Theorem Example (2)

- Carefully define our variables:
  - Let T = I tested positive
  - Let D = I have the disease
  - We want ℙ(D|T)

$\mathbb{P}(T|D) = 0.95$
$\mathbb{P}(T|D^c) = 0.01$
$\mathbb{P}(D) = 0.005$

$$\mathbb{P}(D|T) = \frac{\mathbb{P}(T|D)\mathbb{P}(D)}{\mathbb{P}(T)} = \frac{\mathbb{P}(T|D)\mathbb{P}(D)}{\mathbb{P}(T|D)\mathbb{P}(D) + \mathbb{P}(T|D^c)\mathbb{P}(D^c)}$$

$$\mathbb{P}(D|T) = \frac{(0.95)(0.005)}{(0.95)(0.005) + (0.01)(0.995)} = 0.32$$

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Independence

- Two events are said to be independent of each other if

$$\mathbb{P}(A|B) = \mathbb{P}(A)$$ ← Can only be used if $\mathbb{P}(B) \neq 0$

- Or, equivalently,

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$$

  - If A is independent of B, then B is independent of A (symmetrical)
  - If A is independent of B, then A is also independent of B^c

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Independent Trials

- If an experiment involves a sequence of independent but identical stages, we say each stage is an independent trial
- Example: A coin flip produces a head with probability p and a tail with probability (1-p)
  - What is the probability of getting the sequence HHTHTT?

$$\mathbb{P}(HHTHTT) = \mathbb{P}(H)\mathbb{P}(H)\mathbb{P}(T)\mathbb{P}(H)\mathbb{P}(T)\,\mathbb{P}(T) = p^3(1-p)^3$$

  - What is the probability of getting exactly 3 heads from six coin tosses? (hint: apply total probability rule)

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Review #6: What have we learned?

- Define "conditional probability"
- What is the total probability rule?
- What is the most common application of Bayes' Rule?
- Define "independence"
- Define "independent trial"

