Online Review Course of
Undergraduate Probability and Statistics

Review Lecture 5
Probability, part 2: Counting

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Discrete Uniform Probability

- Consider a discrete finite sample space of size N
  – The probability law can be completely defined by defining the probability of each outcome
- For a uniform probability (e.g., random selection), the probability of each outcome must be 1/N (called the "classical probability concept")
- Thus, for any event E,

Determining probabilities is now a counting problem

$$\mathbb{P}(E) = \frac{\#\,elements\ in\ E}{N} = \frac{|E|}{|\Omega|}$$

Called the cardinality of the set

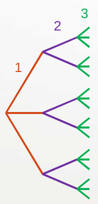
Counting – Combinatorial Analysis

- Counting can be hard
  – How many phone numbers are there in the 512 area code?
  – How many different poker hands are there?
  – How many license plates can they make in Texas?
- We can count by partitioning into cases
  – **Sum Rule**: Given two disjoint sets A and B, picking one element from A or B gives |A| + |B| choices
  – **Product Rule**: Given any two sets A and B, picking one element from A and one element from B gives |A||B| choices

Counting Example

- You choose between doing homework in one of your three classes or going to one of four movies
  – Sum rule: there are 7 options
- You choose between doing homework in one of your three classes and going to one of four movies
  – Product Rule: there are 12 options

Stage Counting Method



- Break the counting problem into r independent stages
  – Stage 1: there are n_1 options
  – Stage 2: there are n_2 options
  – Etc.
- Using the product rule, the total number of options is $n_1\, n_2 \ldots n_r$

Counting Example 1

- How many phone numbers are there in the 512 area code?
  – Stage 1, pick the first number: $n_1 = 8$
  – Stage 2, pick the second number: $n_2 = 10$
  – …
  – Stage 7, pick the last number: $n_7 = 10$
- Multiply all the stage counts:
  – $n_{total} = 8\cdot10\cdot10\cdot10\cdot10\cdot10 = 8$ million

Counting Example 2

- For license plates made with three letters followed by three numbers, how many are possible?
  - $n_{total} = 26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 = 17{,}576{,}000$
  - Note: order matters! ABC123 is a different license plate from CAB321
- This is an example of a string: orderings with repetitions allowed

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Counting Example 3

- For a set with n elements, how many subsets are possible?
  - Stage 1, is element 1 in the subset? $n_1 = 2$
  - Stage 2, is element 2 in the subset? $n_2 = 2$
  - …
- Total number of possible subsets = 2^n

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Counting Example 4

- How many different ways can you line up five people for a photograph?
  - Stage 1, pick a person from the group and put them first in the line: $n_1 = 5$
  - Stage 2, pick the second person for line, $n_2 = 4$
  - Stage 3, pick the third person for line, $n_3 = 3$
  - Etc.
- Multiply all the stage counts:
  - $n_{total} = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5!$
  - This is an example of orderings without repetitions (replacements), also called permutations

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K-Permutations

- How many different ways can you line up 3 people, selected from a group of 8?
  - Stage 1, pick a person from the group and put them first in the line: $n_1 = 8$
  - Stage 2, pick the second person, $n_2 = 7$
  - Stage 3, pick the third person, $n_3 = 6$
  - Multiply all the stage counts: $n_{total} = 8 \cdot 7 \cdot 6 = 8!/5!$
- How many different sequences are possible selecting k out of a set of n?
  - $n_{total} = n!/(n-k)! = {}_nP_k$

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Combinations – When Order Doesn't Matter

- How many different subsets are possible when selecting k elements out of a set of n?
  - Order doesn't matter
  - There is no replacement
  - Define the symbol for this number of combinations (read as "n choose k"): $\binom{n}{k}$
- We can derive the answer by using a two-step derivation of the k-permutations result

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Combinations

- Two-stage derivation of k-permutations
  - Stage 1, pick a k-subset: $n_1 = \binom{n}{k}$
  - Stage 2, order the k elements, $n_2 = k!$
  - Multiply, ${}_nP_k = \binom{n}{k} k!$
- But we already know the answer for the number of k-permutations, so solve for $\binom{n}{k}$:

$$\binom{n}{k} = \frac{{}_nP_k}{k!} = \frac{n!}{k!\,(n-k)!}$$

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Counting Example 5

- How many different 5-card poker hands are there?
  - Drawing without replacement
  - Order doesn't matter
  - n = 52, k = 5
- Calculate "52 choose 5":

$$\binom{52}{5} = \frac{52!}{5!\,(52-5)!} = \frac{52\cdot 51\cdot 50\cdot 49\cdot 48}{5\cdot 4\cdot 3\cdot 2\cdot 1} = 2{,}598{,}960$$

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Combinatorics

- Draw k from a set of n with replacement
  - Order matters: simple product rule = n^k
  - Order doesn't matter: $n^k/k!$
- Draw k from a set of n without replacement
  - Order matters: k-permutations = n!/(n-k)!
  - Order doesn't matter: Combinations = $\binom{n}{k}$
- General problem: break it into stages where each stage is one of the above types

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Review #5: What have we learned?

- Under what circumstances does a probability law turn into merely a counting problem?
- Define the sum rule and the product rule
- What is the stage counting method?
- What is the difference between a permutation and a combination?
- What does "drawing with replacement" mean?

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