ECE431 Homework 3
Sampling and DSP Systems

Due in class Friday, September 21. Submit in WisCEL 410B to the ECE431 lock box.

3.1. Digital Differentiator. The goal is to design a DT filter that approximates the CT differentiation operation. Assume input $x(t)$, a desired output $\frac{dx(t)}{dt}$ and ideal sampling/reconstruction. Hence, the DT differentiator implementation involves sampling $x(t)$, processing the samples with a DT filter having impulse response $g[n]$, and then reconstructing a CT signal from the filter output

a) Differentiation is an LTI system. Let $H(\Omega)$ be the frequency response of the differentiation operation. Find $H(\Omega)$ by comparing the FTs of $x(t)$ and $\frac{dx(t)}{dt}$. $H(\Omega)$ is the desired frequency response we wish to approximate.

b) The derivative is defined as

$$\frac{dx(t)}{dt} = \lim_{h \to 0} \frac{x(t) - x(t-h)}{h}$$

This suggests the DT approximation

$$\frac{dx(t)}{dt} \approx \frac{x[n] - x[n-1]}{T} = \frac{x(nT) - x((n-1)T)}{T}$$

Show that we can implement this approximation in DT with an LTI filter with impulse response $g[n] = \frac{\delta[n]-\delta[n-1]}{T}$. Use MATLAB to plot the magnitude and phase response of this filter.

c) Compare the magnitude and phase of $G(e^{j\omega})$ to $H(\Omega)$.

d) Now consider another DT approximation to differentiation obtained by a system with impulse response

$$f[n] = \frac{\delta[n+1] - \delta[n-1]}{2T}$$

Compare the magnitude and phase response of this filter to the desired filter $H(\Omega)$. Does this filter's phase match the desired phase better than $G(e^{j\omega})$?

e) If $x(t)$ is bandlimited to ± 1 kHz, how fast should we sample to guarantee that $f[n]$ implements a reasonable approximation to the CT differentiator?

3.2. OS 4.31. (Note this is identical to problem 4.25 in the second edition of Oppenheim and Schafer.)

3.3. See Example 4.13 from Section 4.6.3 of Haykin and Van Veen (attached). Note that this text uses ω for CT frequency, while in OS and in class we use ω for DT frequency.

a) Find the design constraints on the analog anti-imaging filter to satisfy the constraint on the overall zero-order hold reconstruction system given in Example 4.13 assuming four-times oversampling. Compare four-times oversampling to no oversampling and eight-times oversampling. You may use MATLAB to plot your design constraints.

b) Now consider the impact of oversampling on the analog anti-aliasing filter. Assume that the magnitude response in the band of interest (± 20 kHz) must lie between 1.01 and 0.99, and that any components that might alias into the band of interest must be attenuated by a factor of 10^4, i.e., the gain must be less than 10^{-4}. Find the constraints on this filter for the standard 44.1 kHz sampling rate and for eight-times oversampling, i.e., a sampling rate of 8(44.1) kHz.

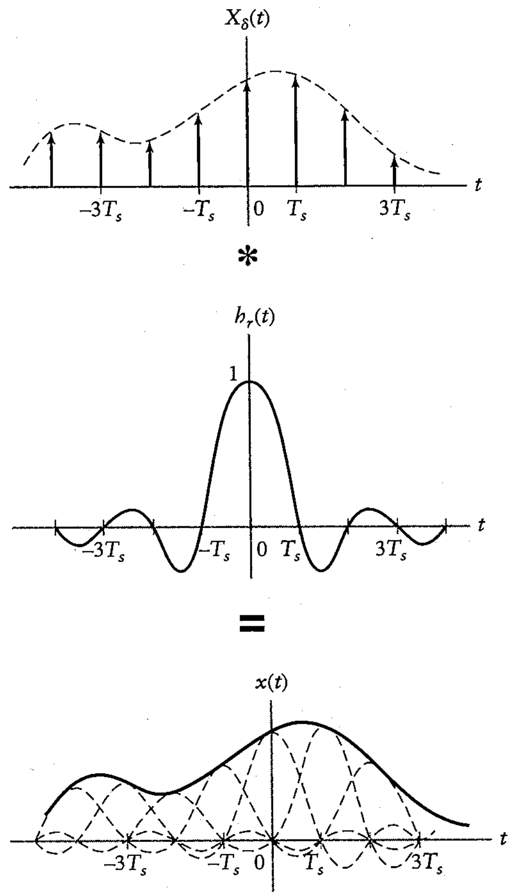

FIGURE 4.36 Ideal reconstruction in the time domain.

signal. In practice, Eq. (4.30) cannot be implemented, for two reasons: First of all, it represents a noncausal system, because the output, $x(t)$, depends on past and future values of the input, $x[n]$; second, the influence of each sample extends over an infinite amount of time, because $h_r(t)$ has infinite duration.

■ 4.6.3 A Practical Reconstruction: The Zero-Order Hold

In practice, a continuous-time signal is often reconstructed by means of a device known as a *zero-order hold*, which simply maintains or holds the value $x[n]$ for T_s seconds, as depicted in Fig. 4.37. This causes sharp transitions in $x_o(t)$ at integer multiples of T_s and produces

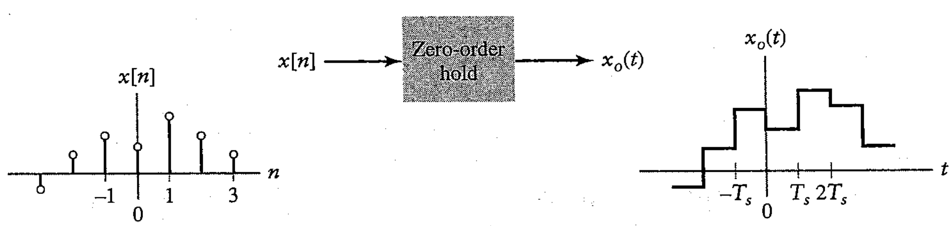

FIGURE 4.37 Reconstruction via a zero-order hold.

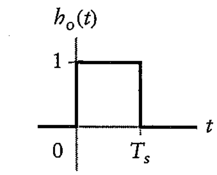

FIGURE 4.38 Rectangular pulse used to analyze zero-order hold reconstruction.

a stair-step approximation to the continuous-time signal. Once again, the FT offers a means for analyzing the quality of this approximation.

The zero-order hold is represented mathematically as a weighted sum of rectangular pulses shifted by integer multiples of the sampling interval. Let

$$h_o(t) = \begin{cases} 1, & 0 < t < T_s \\ 0, & t < 0, t > T_s \end{cases},$$

as depicted in Fig. 4.38. The output of the zero-order hold is expressed in terms of $h_o(t)$ as

$$x_o(t) = \sum_{n=-\infty}^{\infty} x[n]h_o(t - nT_s). \tag{4.31}$$

We recognize Eq. (4.31) as the convolution of the impulse-sampled signal $x_\delta(t)$ with $h_o(t)$:

$$\begin{aligned} x_o(t) &= h_o(t) * \sum_{n=-\infty}^{\infty} x[n]\delta(t - nT_s) \\ &= h_o(t) * x_\delta(t). \end{aligned}$$

Now we take the FT of $x_o(t)$, using the convolution–multiplication property of the FT to obtain

$$X_o(j\omega) = H_o(j\omega)X_\delta(j\omega),$$

from which, on the basis of the result of Example 3.25 and the FT time-shift property, we obtain

$$h_o(t) \xleftrightarrow{FT} H_o(j\omega) = 2e^{-j\omega T_s/2}\frac{\sin(\omega T_s/2)}{\omega}.$$

Figure 4.39 depicts the effect of the zero-order hold in the frequency domain, assuming that T_s is chosen to satisfy the sampling theorem. Comparing $X_o(j\omega)$ with $X(j\omega)$, we see that the zero-order hold introduces three forms of modification:

1. A linear phase shift corresponding to a time delay of $T_s/2$ seconds.
2. A distortion of the portion of $X_\delta(j\omega)$ between $-\omega_m$ and ω_m. [The distortion is produced by the curvature of the mainlobe of $H_o(j\omega)$.]
3. Distorted and attenuated versions of the images of $X(j\omega)$, centered at nonzero multiples of ω_s.

By holding each value $x[n]$ for T_s seconds, we introduce a time shift of $T_s/2$ seconds into $x_o(t)$. This is the source of modification 1. Modifications 2 and 3 are associated with the stair-step approximation. Note that the sharp transitions in $x_o(t)$ suggest the presence of high-frequency components and are consistent with modification 3. Both modifications 1 and 2 are reduced by increasing ω_s or, equivalently, decreasing T_s.

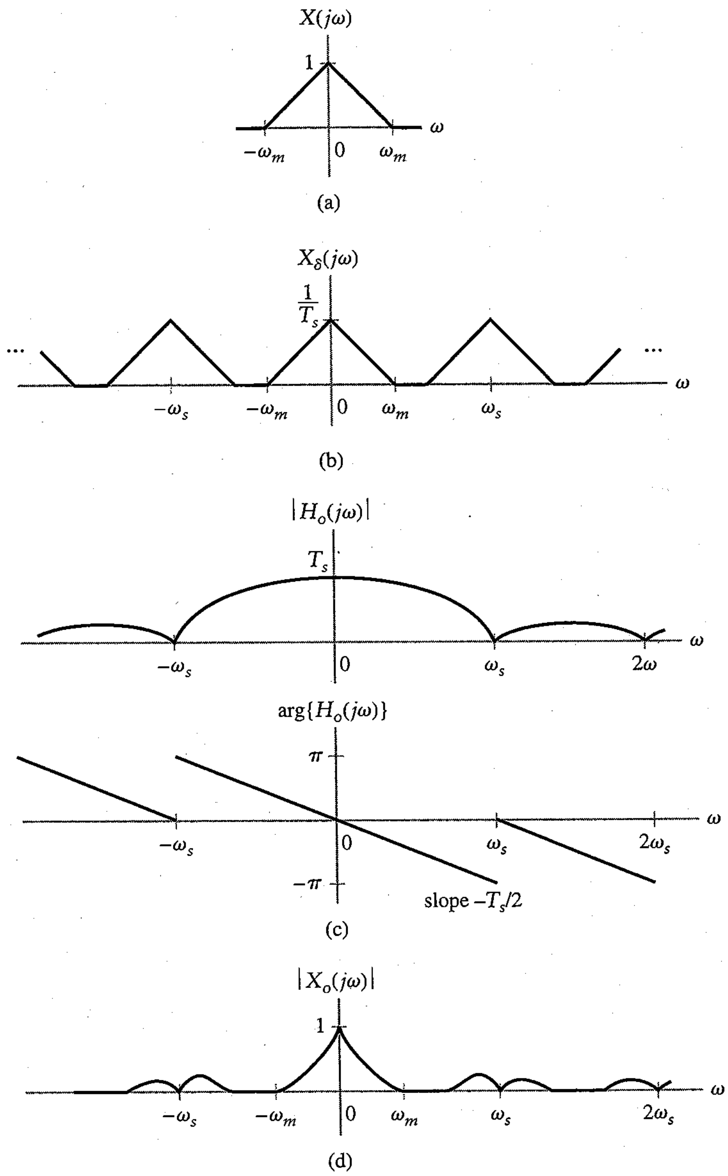

FIGURE 4.39 Effect of the zero-order hold in the frequency domain. (a) Spectrum of original continuous-time signal. (b) FT of sampled signal. (c) Magnitude and phase of $H_o(j\omega)$. (d) Magnitude spectrum of signal reconstructed using zero-order hold.

In some applications, the modifications associated with the zero-order hold may be acceptable. In others, further processing of $x_o(t)$ may be desirable to reduce the distortion associated with modifications 2 and 3. In most situations, a delay of $T_s/2$ seconds is of no real consequence. Modifications 2 and 3 may be eliminated by passing $x_o(t)$ through a continuous-time compensation filter with frequency response

$$H_c(j\omega) = \begin{cases} \dfrac{\omega T_s}{2\sin(\omega T_s/2)}, & |\omega| < \omega_m \\ 0, & |\omega| > \omega_s - \omega_m \end{cases}.$$

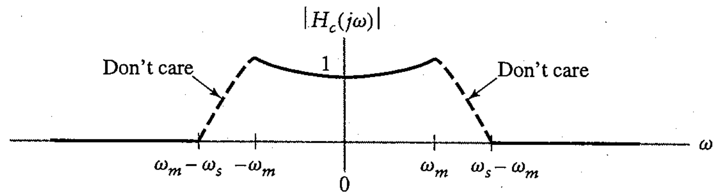

FIGURE 4.40 Frequency response of a compensation filter used to eliminate some of the distortion introduced by the zero-order hold.

The magnitude of this frequency response is depicted in Fig. 4.40. On $|\omega| < \omega_m$, the compensation filter reverses the distortion introduced by the mainlobe curvature of $H_o(j\omega)$. For $|\omega| > \omega_s - \omega_m$, $H_c(j\omega)$ removes the energy in $X_o(j\omega)$ centered at nonzero multiples of ω_s. The value of $H_c(j\omega)$ does not matter on the frequency band $\omega_m < |\omega| < \omega_s - \omega_m$, since $X_o(j\omega)$ is zero there. $H_c(j\omega)$ is often termed an *anti-imaging filter*, because it eliminates the distorted images of $X(j\omega)$ that are present at nonzero multiples of ω_s. A block diagram representing the compensated zero-order hold reconstruction process is depicted in Fig. 4.41. The anti-imaging filter smooths out the step discontinuities in $x_o(t)$.

Several practical issues arise in designing and building an anti-imaging filter. We cannot obtain a causal anti-imaging filter that has zero phase; hence a practical filter will introduce some phase distortion. In many cases, a linear phase in the passband, $|\omega| < \omega_m$, is acceptable, since linear-phase distortion corresponds to an additional time delay. The difficulty of approximating $|H_c(j\omega)|$ depends on the separation between ω_m and $\omega_s - \omega_m$. First of all, if this distance, $\omega_s - 2\omega_m$, is large, then the mainlobe curvature of $H_o(j\omega)$ is very small, and a good approximation is obtained simply by setting $|H_c(j\omega)| = 1$. Second, the region $\omega_m < \omega < \omega_s - \omega_m$ is used to make the transition from passband to stopband. If $\omega_s - 2\omega_m$ is large, then the transition band of the filter is large. Filters with large transition bands are much easier to design and build than those with small transition bands. Hence, the requirements on an anti-imaging filter are greatly reduced by choosing T_s sufficiently small so that $\omega_s \gg 2\omega_m$. (A more detailed discussion of filter design is given in Chapter 8.)

In practical reconstruction schemes, it is common to increase the effective sampling rate of the discrete-time signal prior to the zero-order hold. This technique, known as *oversampling*, is done to relax the requirements on the anti-imaging filter, as illustrated in the next example. Although doing so increases the complexity of the discrete-time hardware, it usually produces a decrease in overall system cost for a given level of reconstruction quality.

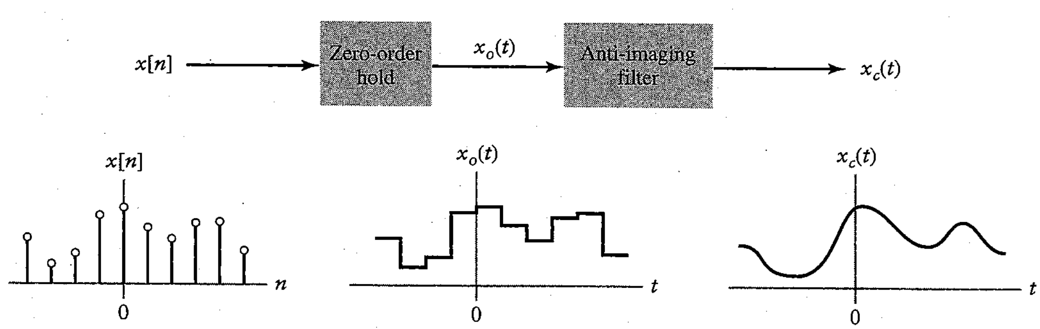

FIGURE 4.41 Block diagram of a practical reconstruction system.

EXAMPLE 4.13 OVERSAMPLING IN CD PLAYERS In this example, we explore the benefits of oversampling in reconstructing a continuous-time audio signal using an audio compact disc player. Assume that the maximum signal frequency is $f_m = 20$ kHz. Consider two cases: (a) reconstruction using the standard digital audio rate of $1/T_{s1} = 44.1$ kHz, and (b) reconstruction using eight-times oversampling, for an effective sampling rate of $1/T_{s2} = 352.8$ kHz. In each case, determine the constraints on the magnitude response of an anti-imaging filter so that the overall magnitude response of the zero-order hold reconstruction system is between 0.99 and 1.01 in the signal passband and the images of the original signal's spectrum centered at multiples of the sampling frequency [the $k = \pm 1, \pm 2, \ldots$ terms in Eq. (4.23)] are attenuated by a factor of 10^{-3} or more.

Solution: In this example, it is convenient to express frequency in units of hertz rather than radians per second. This is explicitly indicated by replacing ω with f and by representing the frequency responses $H_o(j\omega)$ and $H_c(j\omega)$ as $H'_o(jf)$ and $H'_c(jf)$, respectively. The overall magnitude response of the zero-order hold followed by an anti-imaging filter $H'_c(jf)$ is $|H'_o(jf)||H'_c(jf)|$. Our goal is to find the acceptable range of $|H'_c(jf)|$ so that the product $|H'_o(jf)||H'_c(jf)|$ satisfies the constraints on the response. Figures 4.42(a) and (b) depict $|H'_o(jf)|$, assuming sampling rates of 44.1 kHz and 352.8 kHz, respectively. The dashed lines in each figure denote the signal passband and its images. At the lower sampling rate [Fig. 4.42(a)], we see that the signal and its images occupy the majority of the spectrum; they are separated by 4.1 kHz. In the eight-times oversampling case [Fig. 4.42(b)], the signal and its images occupy a very small portion of the much wider spectrum; they are separated by 312.8 kHz.

The passband constraint is $0.99 < |H'_o(jf)||H'_c(jf)| < 1.01$, which implies that

$$\frac{0.99}{|H'_o(jf)|} < |H'_c(jf)| < \frac{1.01}{|H'_o(jf)|}, \quad -20\,\text{kHz} < f < 20\,\text{kHz}.$$

Figure 4.42(c) depicts these constraints for both cases. Here, we have multiplied $|H'_c(jf)|$ by the sampling interval T_{s1} or T_{s2}, so that both cases are displayed with the same vertical scale. Note that case (a) requires substantial curvature in $|H'_c(jf)|$ to eliminate the passband distortion introduced by the mainlobe of $H'_o(jf)$. At the edge of the passband, the bounds are as follows:

Case (a):

$$1.4257 < T_{s1}|H'_c(jf_m)| < 1.4545, \quad f_m = 20\,\text{kHz}$$

Case (b):

$$0.9953 < T_{s2}|H'_c(jf_m)| < 1.0154, \quad f_m = 20\,\text{kHz}$$

The image-rejection constraint implies that $|H'_o(jf)||H'_c(jf)| < 10^{-3}$ for all frequencies at which images are present. This condition is simplified somewhat by considering only the frequency at which $|H'_o(jf)|$ is largest. The maximum value of $|H'_o(jf)|$ in the image frequency bands occurs at the smallest frequency in the first image: 24.1 kHz in case (a) and 332.8 kHz in case (b). The value of $|H'_o(jf)|/T_{s1}$ and $|H'_o(jf)|/T_{s2}$ at these frequencies is 0.5763 and 0.0598, respectively, which implies that the bounds are

$$T_{s1}|H'_c(jf)| < 0.0017, \quad f > 24.1\,\text{kHz},$$

and

$$T_{s2}|H'_c(jf)| < 0.0167, \quad f > 332.8\,\text{kHz},$$

for cases (a) and (b), respectively. Hence, the anti-imaging filter for case (a) must show a transition from a value of $1.4257/T_{s1}$ to $0.0017/T_{s1}$ over an interval of 4.1 kHz. In contrast,

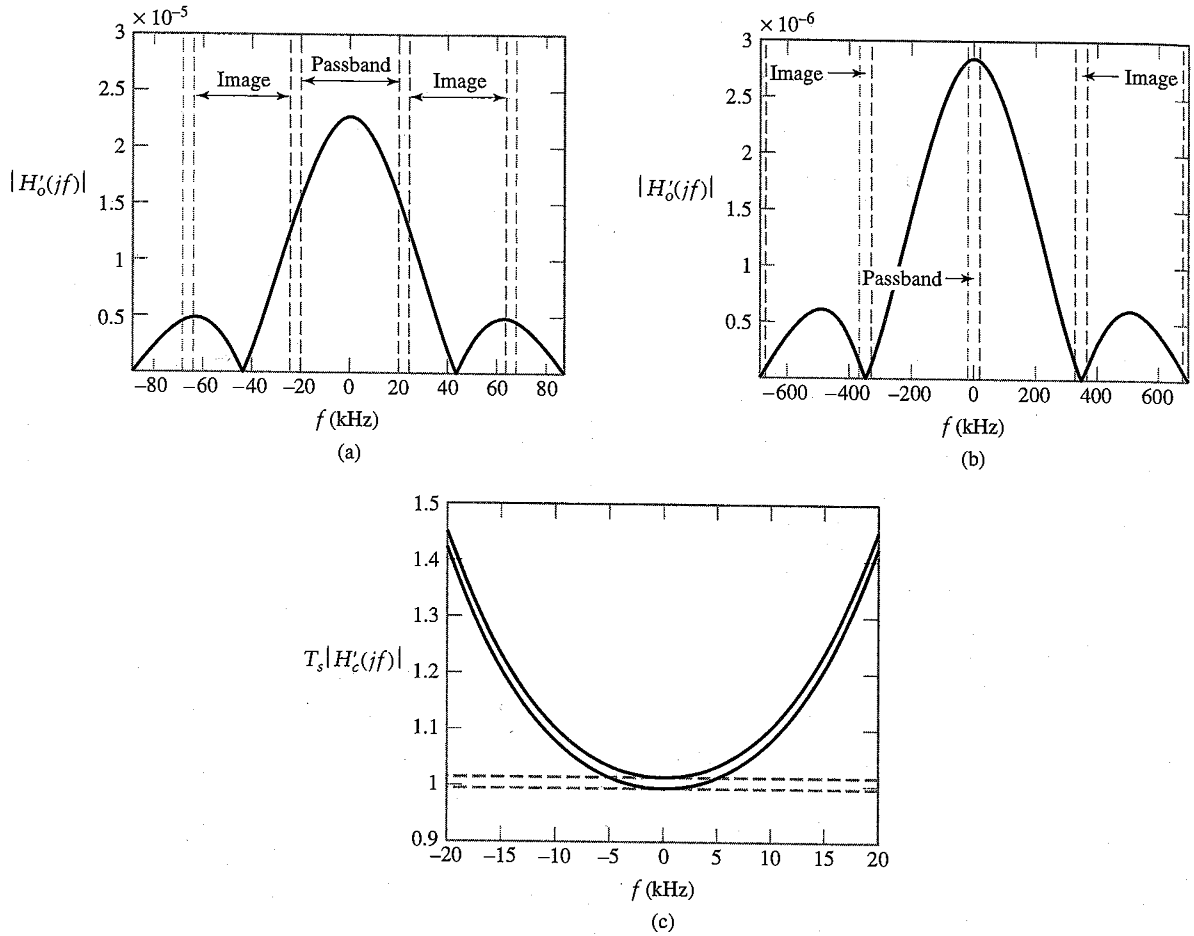

FIGURE 4.42 Anti-imaging filter design with and without oversampling. (a) Magnitude of $H_o'(jf)$ for 44.1-kHz sampling rate. Dashed lines denote signal passband and images. (b) Magnitude of $H_o'(jf)$ for eight-times oversampling (352.8-kHz sampling rate). Dashed lines denote signal passband and images. (c) Normalized constraints on passband response of anti-imaging filter. Solid lines assume a 44.1-kHz sampling rate; dashed lines assume eight-times oversampling. The normalized filter response must lie between each pair of lines.

with eight-times oversampling the filter must show a transition from $0.9953/T_{s2}$ to $0.0167/T_{s2}$ over a frequency interval of 312.8 kHz. Thus, oversampling not only increases the transition width by a factor of almost 80, but also relaxes the stopband attenuation constraint by a factor of more than 10. ■

4.7 *Discrete-Time Processing of Continuous-Time Signals*

In this section, we use Fourier methods to discuss and analyze a typical system for the discrete-time processing of continuous-time signals. There are several advantages to processing a continuous-time signal with a discrete-time system. These advantages result from the power and flexibility of discrete-time computing devices. First, a broad class of signal