ECE 431 Homework 2

Solutions

2.1 a) Show that DTFT of $x[n] * y[n]$ is $X(e^{j\omega})Y(e^{j\omega})$

Starting w/ Definition

$$\text{DTFT}\left[x[n] * y[n]\right] = \sum_{n=-\infty}^{\infty} (x[n] * y[n])\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x[k]\, y[n-k]\, e^{-j\omega n}$$

$$= \sum_{k=-\infty}^{\infty} x[k] \sum_{n=-\infty}^{\infty} y[n-k]\, e^{-j\omega n}$$

let $m = n - k$

$$= \sum_{k=-\infty}^{\infty} x[k] \sum_{m=-\infty}^{\infty} y[m]\, e^{-j\omega(m+k)}$$

$$= \sum_{k=-\infty}^{\infty} x[k]\, e^{-j\omega k} \sum_{m=-\infty}^{\infty} y[m]\, e^{-j\omega m}$$

$$= X(e^{j\omega})\, Y(e^{j\omega})$$

b) Suppose $x(t)$ is real. What can you conclude about FT?

Recall:

$$X(j\Omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\Omega t}\, dt$$

thus

$$X^*(j\Omega) = \int_{-\infty}^{\infty} x(t)\, e^{j\Omega t}\, dt \quad \text{if } x(t) \text{ is real}$$

$$X^*(j\Omega) = X(-j\Omega)$$

thus,

1) real part of $X(j\Omega)$ is even

$$X_R(j\Omega) = X_R(-j\Omega)$$

2) Imaginary part is odd

$$X_I(j\Omega) = -X_I(-j\Omega)$$

3) Magnitude is even

$$|X(j\Omega)| = |X(-j\Omega)|$$

4) Phase is odd

$$\angle X(j\Omega) = -\angle X(-j\Omega)$$

c) Prove $w(t) * (x(t) + y(t)) = w(t) * x(t) + w(t) * y(t)$

From definition

$$w(t) * (x(t) + y(t)) = \int_{-\infty}^{\infty} w(\tau)\big(x(t-\tau) + y(t-\tau)\big)\, d\tau$$

$$= \int_{-\infty}^{\infty} w(\tau)\, x(t-\tau)\, d\tau + \int_{-\infty}^{\infty} w(\tau)\, y(t-\tau)\, d\tau$$

$$= w(t) * x(t) + w(t) * y(t)$$

d) Show

$$x[n] \longrightarrow \boxed{g[n]} \longrightarrow \boxed{h[n]} \longrightarrow y_1[n]$$

and

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow \boxed{g[n]} \longrightarrow y_2[n]$$

then

$$y_1[n] = y_2[n]$$

2 Ways

- From part a) write

$$\begin{aligned} Y_1(e^{j\omega}) &= X(e^{j\omega})\, G(e^{j\omega})\, H(e^{j\omega}) \quad \text{clearly} \\ &= X(e^{j\omega})\, H(e^{j\omega})\, G(e^{j\omega}) \\ &= Y_2(e^{j\omega}) \end{aligned}$$

so

$$y_1[n] = y_2[n]$$

- Or, write out convolution.

2.2

Consider continuous time signal

$$x(t) = \frac{\sin \pi t}{\pi t} \cdot \sin 9\pi t$$

Sketch the FT of $X_s(\Omega)$ of $x_s(t) = \sum_{n=-\infty}^{\infty} x(nT)\,\delta(t-nT)$

and the DTFT $X(e^{j\omega})$ of $x[n] = x(nT)$ for:

a) $T = 1/14$

First, just sketch FT of $x(t)$.

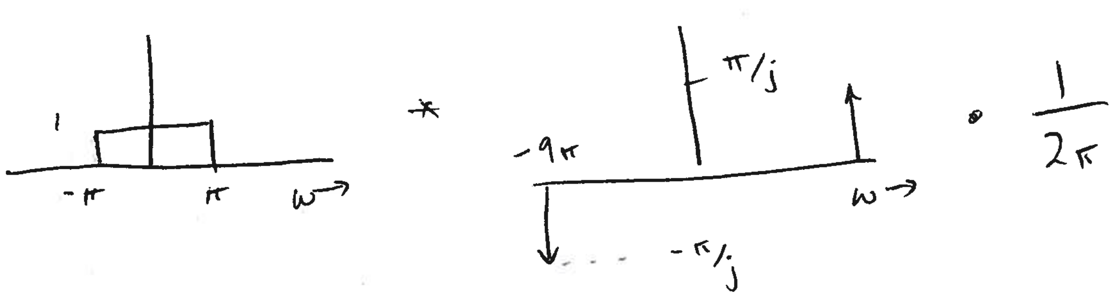
so $FT\left(\frac{\sin \pi t}{\pi t}\right) = \begin{cases} 1 & |\Omega| \le \pi \\ 0 & \text{else} \end{cases}$

$$FT(\sin 9\pi t) = \frac{\pi}{j}\,\delta(\Omega - 9\pi) - \frac{\pi}{j}\,\delta(\Omega + 9\pi)$$

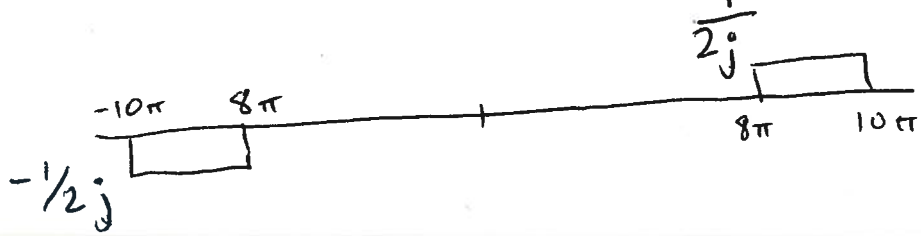
So $X(j\Omega) = \frac{1}{2\pi} FT\left(\frac{\sin \pi t}{\pi t}\right) * FT(\sin 9\pi t)$

$$= \begin{cases} \frac{1}{2j} & 8\pi \le \Omega \le 10\pi \\ -\frac{1}{2j} & -10\pi \le \Omega \le -8\pi \\ & \text{else} \end{cases}$$

pic



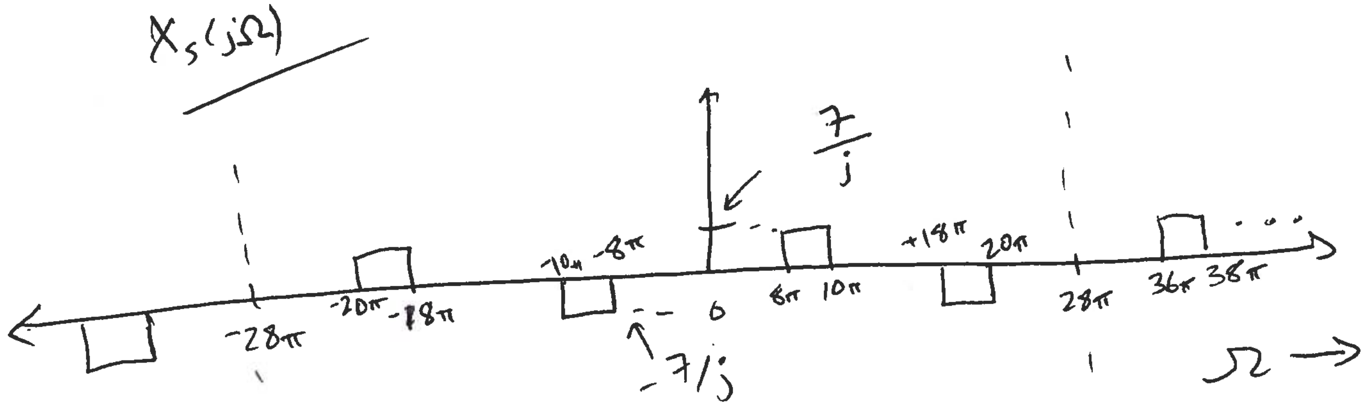
$X(j\Omega) =$

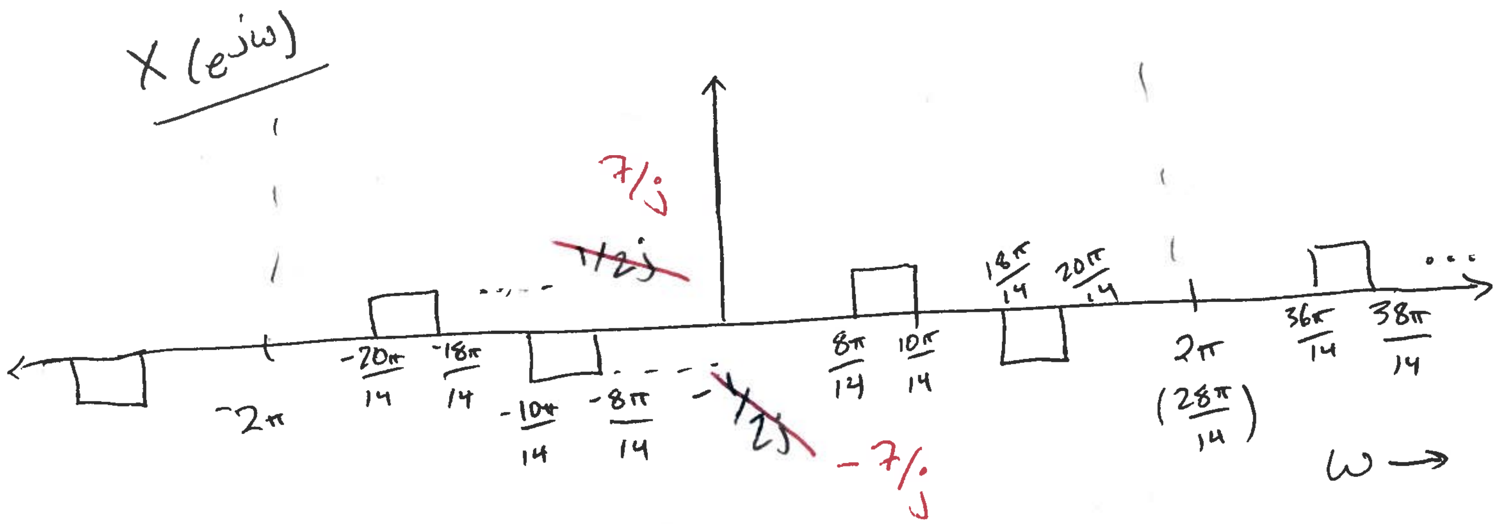


a/ continued

$$X_s(j\Omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X(j(\Omega - k\Omega_s))$$

$\Omega_s = \frac{2\pi}{T} = 28\pi$ $\rightarrow$ so, just have copies of $X(j\Omega)$ every 28π.

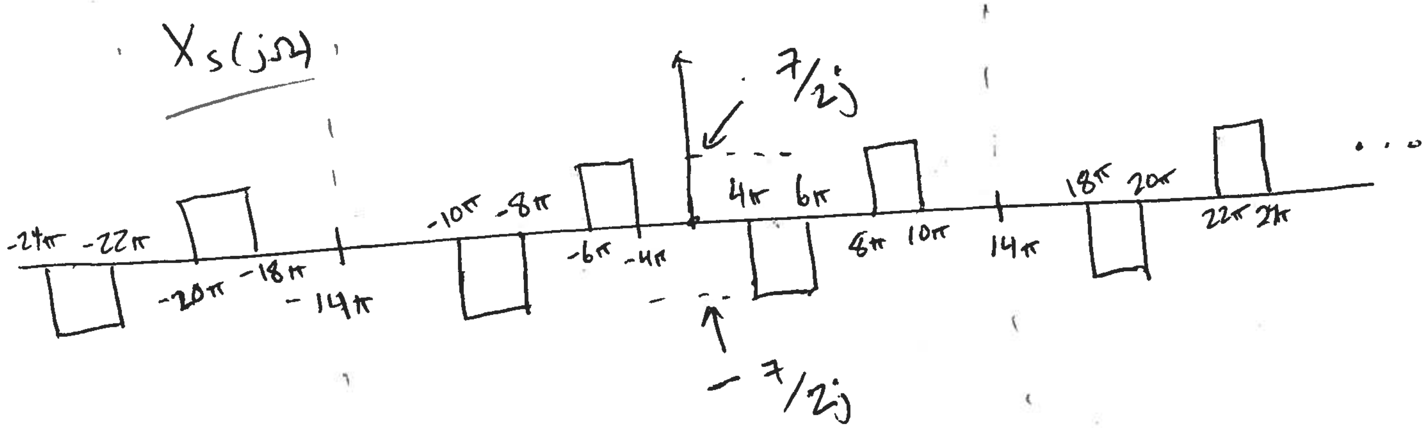


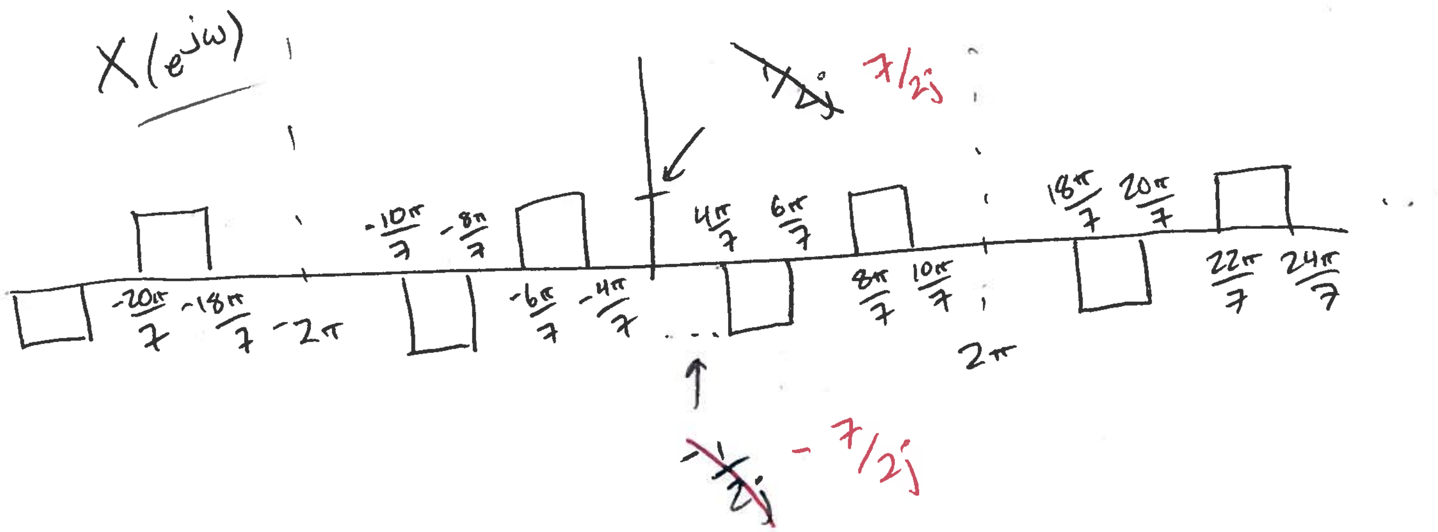


No Aliasing

5b

$$T = 1/7 \Rightarrow \Omega_s = \frac{2\pi}{T} = 14\pi$$

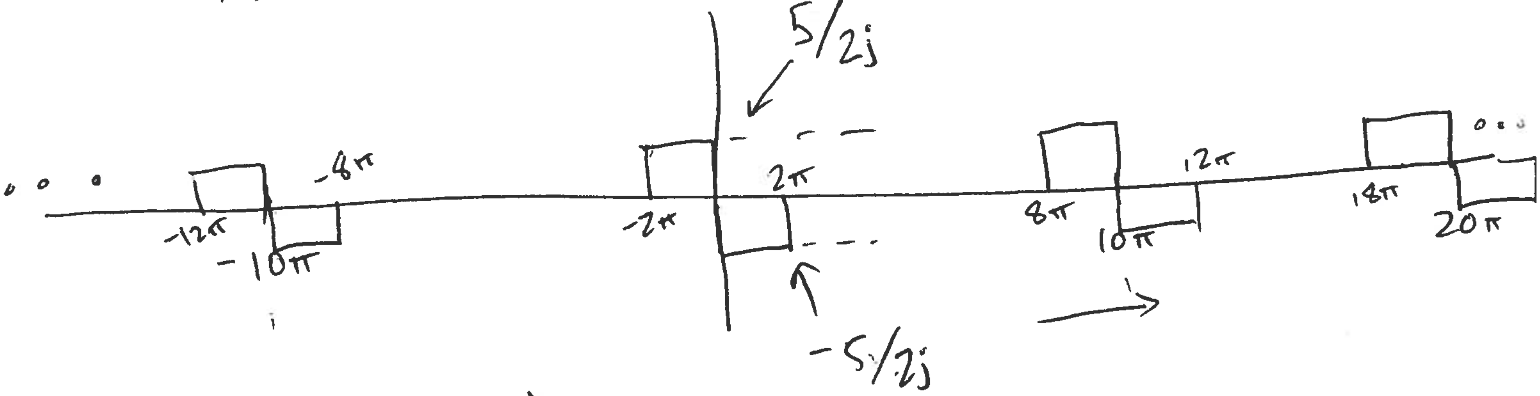




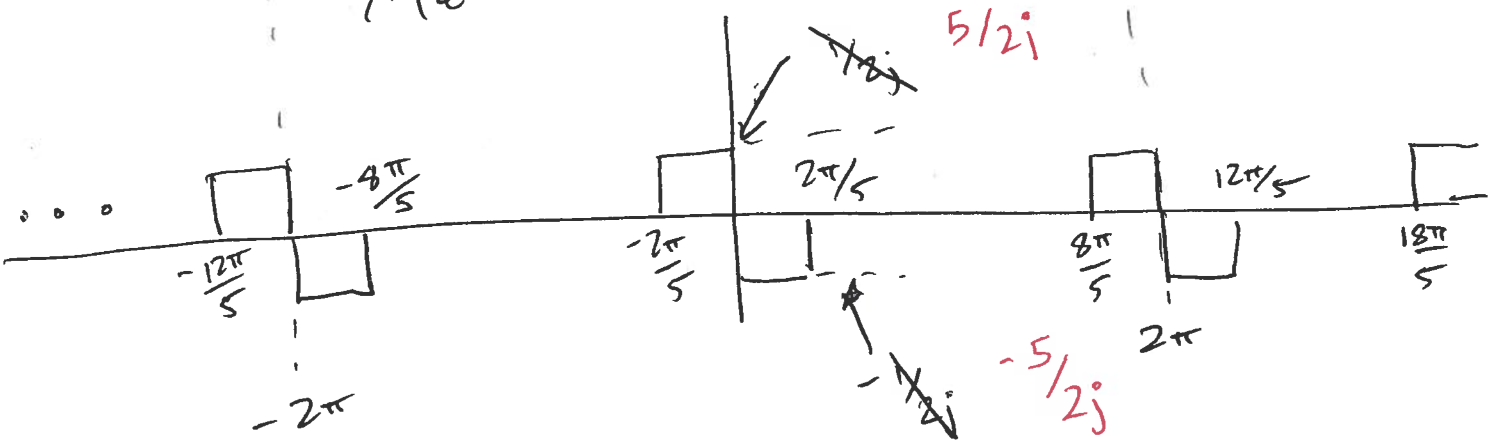
Yes aliasing occurs – but signals don't actually overlap, so we could recover continuous time signal with a-priori knowledge.

c/ $T = 1/5 \Rightarrow \quad {}_s = 10\pi$

$X_s(j\Omega)$



$X(e^{j\omega})$



Again, aliasing occurs, but we could completely recover C.t. signal with some a-priori knowledge.

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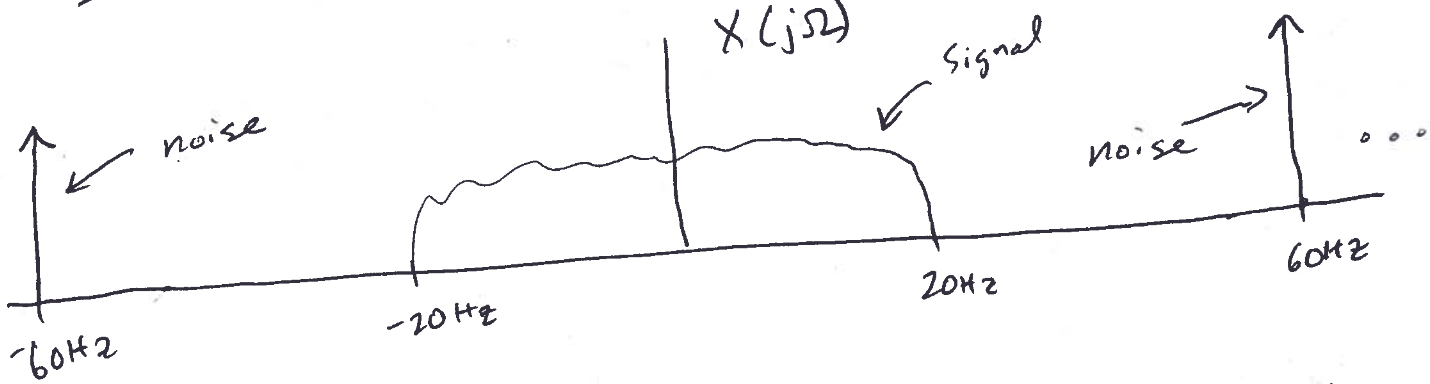
2.3 EKG → first, draw signal in freq.



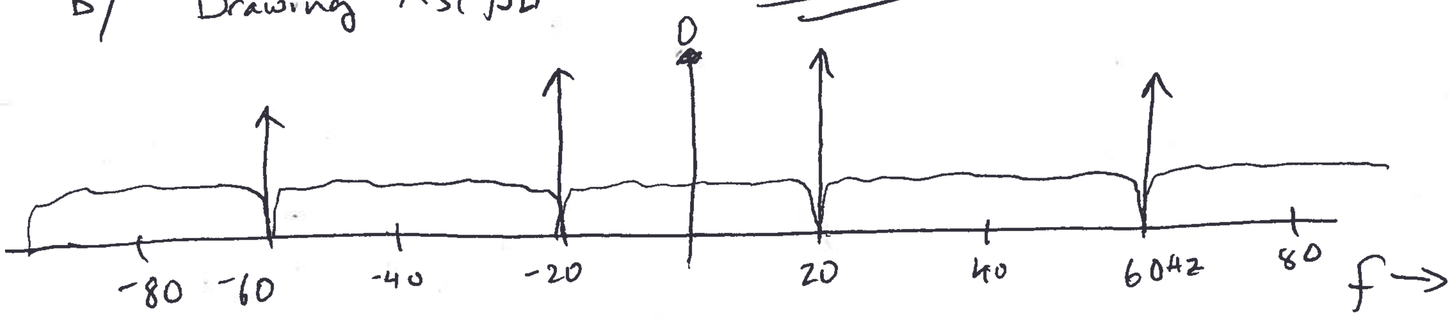
a/ since our max signal frequency is 60 Hz (using notation from book)

$\Omega_N = 2\pi f = 120\pi \rightarrow \Omega_s \geq 2\Omega_N = 240\pi$

So, Ω_s must be greater than $\boxed{240\pi}$.

Also, just can say we must sample at twice max frequency, so $2 \cdot f_{max} =$ 120 samples per sec.

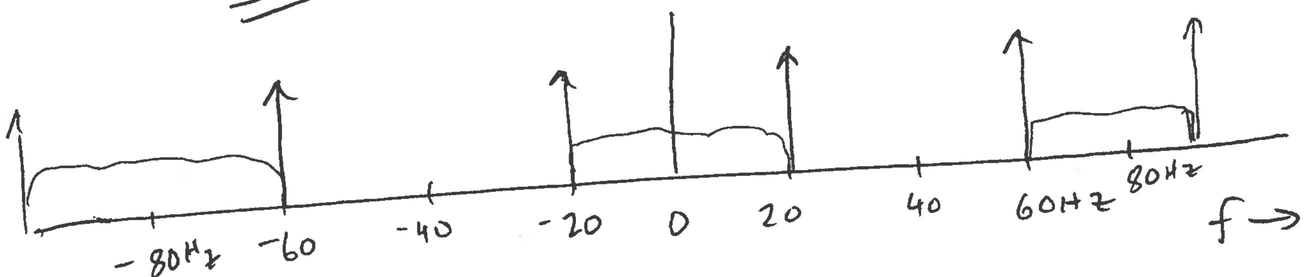
b/ Drawing $X_s(j\Omega)$ 2 answers



From picture, could sample at just 40 samples per second, or $\boxed{80\pi = \Omega_s}$ if we can tolerate losing EKG signal right at 20Hz. (Next)

2.3 continued

b/ OR $X_s(j\Omega)$

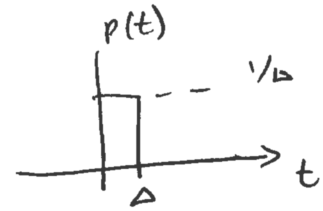


If we need signal at exactly 20 Hz also, then, we must sample at > 80 samples per second, or $\boxed{\Omega_s = 160\pi}$

→ we can tolerate aliasing since we know there are open frequencies over our bandwidth.

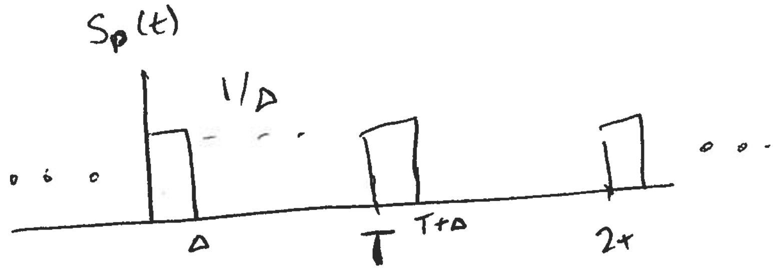
2.4

$$p(t) = \begin{cases} 0 & t < 0 \\ 1/\Delta & 0 < t < \Delta \\ 0 & t > \Delta \end{cases}$$



$T > \Delta$

$$X_p = x(t)\, S_p(t) = x(t) \sum_{n=-\infty}^{\infty} p(t-nT)$$



So, $S_p(t) = p(t) * \sum_{n=-\infty}^{\infty} \delta(t-nT)$

Thus

$$X_p(t) = x(t) \cdot p(t) * \sum_{n=-\infty}^{\infty} \delta(t-nT)$$

$$X_p(\Omega) = \frac{1}{2\pi} X(\Omega) * \left[P(\Omega) \cdot \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\Omega - k\Omega_s) \right]$$

$$P(\Omega) = \frac{2 \sin \Omega \Delta/2}{\Delta \Omega} e^{-j\Omega\Delta/2}$$

So finally

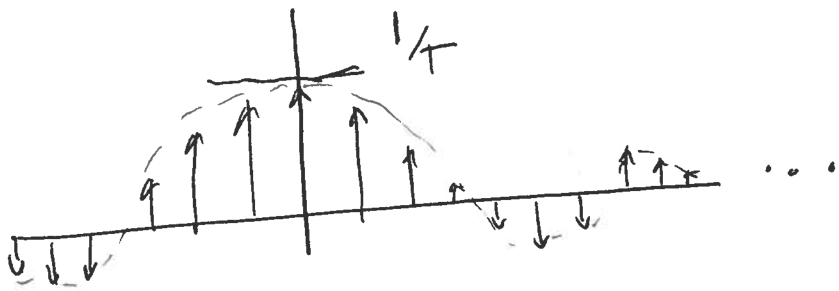
$$X_p(\Omega) = \frac{X(\Omega)}{2\pi} * \underbrace{\frac{2 \cdot \sin \Omega \Delta/2}{\Delta \Omega} e^{-j\Omega\Delta/2} \cdot \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta\left(\Omega - k\frac{2\pi}{T}\right)}_{\text{Sampled Sinc fcn}}$$

(OVER)

Picture

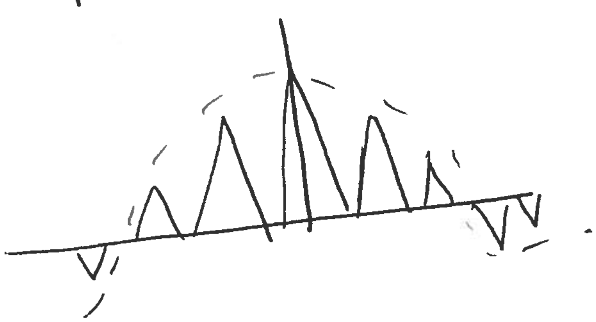
$$X_p(\Omega) = X(\Omega) * S_p(\Omega) = X(\Omega) *$$

$S_p(\Omega)$ (sampled sinc fcn) $\frac{1}{T} \frac{2 \sin \Omega \Delta/2}{\Delta \Omega} \cdot \sum_{k=-\infty}^{\infty} \delta\left(\Omega - k\frac{2\pi}{T}\right)$



Then, if $X(\Omega)$ is band limited

$X_p(\Omega)$



and we can recover signal.