Solutions

ECE431 Homework 1

Thursday, September 10, 2009

1. (OSB 2.30)

(a) Determine and sketch $w[n]$ if $x[n] = (-1)^n u[n]$. Also, determine the overall output $y[n]$.

Solution: The convolutions are most easily computed by drawing out the graphs. We can also compute the output directly from the eqnarrays. We write the first impulse response as

$$h_1[n] = u[n] - u[n-4].$$

The output after the first stage will be

$$\begin{aligned}
w[n] &= h_1[n] * x[n] \\
&= \sum_{k=-\infty}^{\infty} h_1[k]x[n-k] \\
&= \sum_{k=-\infty}^{\infty} \left(u[k] - u[k-4]\right)(-1)^{n-k}u[n-k] \\
&= \sum_{k=-\infty}^{n} \left(u[k] - u[k-4]\right)(-1)^{n-k} \\
&= \sum_{k=0}^{\min(n,3)} (-1)^{n-k} \\
&= \left\{ \begin{array}{cc} 1 & n = 0, 2 \\ 0 & \text{else} \end{array} \right\} \\
&= \delta(n) + \delta(n-2)
\end{aligned}$$

The impulse response of the second system is

$$h_2[n] = u[n+3] - u[n-1].$$

We can write the output of the second stage as

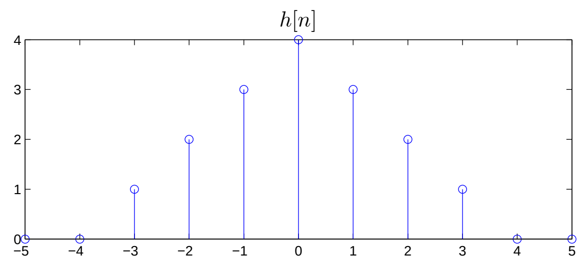
$$\begin{aligned}
y[n] &= h_2[n] * w[n] \\
&= (u[n+3] - u[n-1]) * (\delta(n) + \delta(n-2)) \\
&= (u[n+3] - u[n-1]) * \delta(n) + (u[n+3] - u[n-1]) * \delta(n-2) \\
&= u[n+3] - u[n-1] + u[n+1] - u[n-3])
\end{aligned}$$

$$= \left\{ \begin{array}{cc} 1 & n = -3, -2, 1, 2 \\ 2 & n = -1, 0 \\ 0 & \text{else} \end{array} \right\}$$

(b) Determine and sketch the overall impulse response of the cascade system; i.e. plot the output $y[n] = h[n]$ when $x[n] = \delta[n]$.

Solution: We have

$$\begin{aligned} h[n] &= h_1[n] * h_2[n] \\ &= (u[n] - u[n-4]) * (u[n+3] - u[n-1]) \\ &= u[n] * u[n+3] - u[n] * u[n-1] - u[n-4] * u[n+3] + u[n-4] * u[n-1] \\ &= nu[n+3] - nu[n-1] - nu[n-1] + nu[n-5] \\ &= n(u[n+3] - 2u[n-1] + u[n-5]) \end{aligned}$$

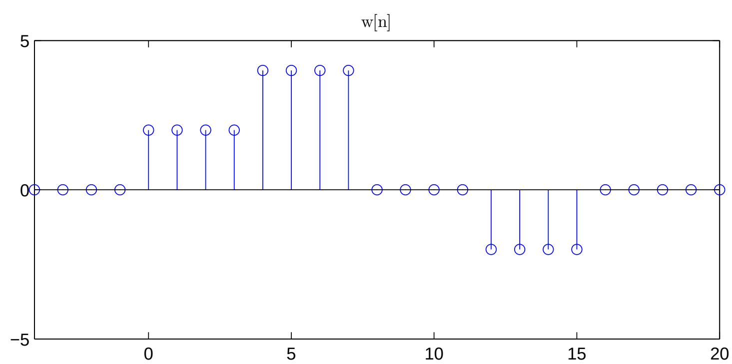


which can also be computed by graphing.

(c) Now consider the input $x[n] = 2\delta[n] + 4\delta[n-4] - 2\delta[n-12]$. Sketch w[n].

Solution: Again, we can write the output of the system as

$$\begin{aligned} w[n] &= h_1[n] * x[n] \\ &= h_1[n] * (2\delta[n] + 4\delta[n-4] - 2\delta[n-12]) \\ &= 2h_1[n] - 4h_1[n-4] - 2h_1[n-12] \end{aligned}$$



Code to convolve two sequences in matlab

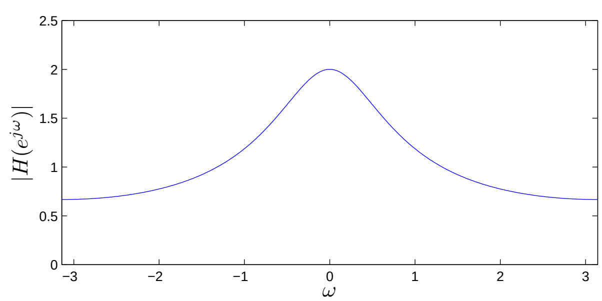
```
N = 40
n = [-N/2:1:N/2];
h1 = [zeros(1,20) 1 1 1 1  zeros(1,17)];
h2 = [zeros(1,17) 1 1 1 1  zeros(1,20)];
x = (-1).^n.*heaviside(n+.01);

%convolve h1 and h2 to get overall impulse response
for m = 1:1:N+1
    w(m)=x*flipud(circshift(h1.',-n(m)));
    %the above line uses x'x which multiplies and sums all elements of a
    %vector
end
w(1:10)=0;
w(length(w)-10:length(w))=0;
%zero padding
```

2. Consider an LTI system whose impulse response is $h[n] = (0.5)^n u[n]$.

(a) Determine the frequency response of the system. Sketch the magnitude of the frequency response. Is the system highpass, bandpass, or lowpass?

Solution: Starting with the definition of the discrete time Fourier transform, we have

$$\begin{aligned} H(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n} \\ &= \sum_{n=-\infty}^{\infty} (0.5)^n u[n]e^{-j\omega n} \\ &= \sum_{n=0}^{\infty}(0.5)^n e^{-j\omega n} \\ &= \sum_{n=0}^{\infty}(0.5e^{-j\omega})^n \\ &= \frac{1}{1-0.5e^{-j\omega}} \end{aligned}$$



The system is low pass.

(b) Let $x[n] = \cos(\pi n) + 4\cos(\frac{\pi}{2}n)$ be the input to this system. Compute the output $y[n] = x[n] * h[n]$.

Solution: We can compute the output of the system in the frequency domain. First,

$$
\begin{aligned}
X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \\
&= \pi\delta(\omega-\pi) + \pi\delta(\omega+\pi) + 4\pi\delta(\omega-\frac{\pi}{2}) + 4\pi\delta(\omega+\frac{\pi}{2}) \qquad (-\pi \le \omega \le \pi)
\end{aligned}
$$

where $X(e^{j\omega})$ is periodic with period 2π. The output of the system is computed as

$$
\begin{aligned}
Y(e^{j\omega}) &= H(e^{j\omega})X(e^{j\omega}) \\
&= \frac{1}{1-0.5e^{-j\omega}}\left(\pi\delta(\omega-\pi) + \pi\delta(\omega+\pi) + 4\pi\delta(\omega-\frac{\pi}{2}) + 4\pi\delta(\omega+\frac{\pi}{2})\right) \\
&= \frac{\pi}{1-0.5e^{-j\pi}}\delta(\omega-\pi) + \frac{\pi}{1-0.5e^{j\pi}}\delta(\omega+\pi) + \\
&\quad \frac{4\pi}{1-0.5e^{-j\pi/2}}\delta(\omega-\frac{\pi}{2}) + \frac{4\pi}{1-0.5e^{j\pi/2}}\delta(\omega+\frac{\pi}{2}) \qquad (-\pi \le \omega \le \pi)
\end{aligned}
$$

Transforming the above equation back to the time domain, we have

$$
y[n] = \frac{2}{3}\cos(\pi n) + \frac{16}{5}\cos(\frac{\pi}{2}n) + \frac{8}{5}\sin(\frac{\pi}{2}n)
$$

This agrees with our intuition - when we input pure sinusoids into an LTI system, the output is a pure sinusoids of the same frequency.

(c) Let $x[n] = \delta[n-1] + 3\delta[n-3]$ be the input to the system. Compute the output $y[n] = x[n] * h[n]$.

Solution: Unlike part (b), the output is most easily computed in the time domain.

$$
\begin{aligned}
y[n] &= x[n] * h[n] \\
&= (\delta[n-1] + 3\delta[n-3]) * 0.5^n u[n] \\
&= 0.5^{n-1}u[n-1]0.5^{n-3}u[n-3] \\
&\cancel{=}\ \cancel{0.5^{2n-4}u[n-3]}
\end{aligned}
$$

3. Download the MATLAB data file mclips.mat and the MATLAB program music.m from the web site. The program will playback either of the two music clips x and y contained in mclips.mat. The clip y is a version of x that has been digitally processed in MATLAB to produce an echo effect.

   (a) Design and implement a linear system in MATLAB that processes the original clip x to produce an echo with a 0.1 second delay (this should sound similar to y). Turn in the derivation of your echo system and the MATLAB code that implements it.

(b) Design and implement a nonlinear system in MATLAB that produces a saturation effect. That is, mimic the effect of pushing an amplifier into saturation by clipping x whenever the magnitude of its amplitude is greater than a certain level (e.g., $|x[n]| > 0.7$). Turn in your design and the MATLAB code that implements it.

Solution: Generated using MATLAB publishing -

0.1 Homework 1 Problem 3

```
close all
clc
clear

load mclips.mat

% plot the audio signal x
plot(x)
```

0.2 Part a) Create Echo

```
yecho = [x; zeros(fs,1)]+ [zeros(fs,1); x];
yecho = yecho/max(yecho);
```

0.3 Part b) Clip Audio

```
yclip = y;
yclip(yclip>0.3) = 0.3;
yclip(yclip<-0.3) = -0.3;
sound(yclip,fs)
plot(yclip)
axis([0 length(yclip) -1 1])
```

4. Verify the inverse DTFT formula

$$h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(\omega)e^{j\omega n}d\omega$$

by replacing $H(\omega)$ with the DTFT expression

$$H(\omega) = \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}$$

and show that you obtain $h[n]$ on both sides of the eqnarray.

Solution: As the problem suggests, combine the two eqnarrays.

$$\begin{aligned} h[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}e^{j\omega n}d\omega \\ &= \frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\int_{-\pi}^{\pi} h[k]e^{-j\omega(k-n)}d\omega \\ &= \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} h[k]\int_{-\pi}^{\pi} e^{-j\omega(k-n)}d\omega \end{aligned}$$

Evaluating the integral, and using Euler's Identity, we have

$$\int_{-\pi}^{\pi} e^{-j\omega(k-n)}d\omega = \frac{e^{-j\pi(k-n)}}{-j(k-n)} - \frac{e^{j\pi(k-n)}}{-j(k-n)} = 2\frac{\sin(\pi(k-n))}{(k-n)}$$

Notice that $2\sin(\pi(k-n))/(k-n)$ is zero for all integer values of $(k-n)$, except when $(k-n)=0$. In this case, since both the numerator and denominator go to zero, we define the answer to be the limit as $(k-n)$ goes to zero. Using l'Hopital's rule

$$\lim_{(k-n)\to 0}\frac{2\sin(\pi(k-n))}{(k-n)} = 2\pi$$

and thus

$$2\frac{\sin(\pi(k-n))}{(k-n)} = 2\pi\delta(k-n).$$

Finally, we have

$$h[n] = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} 2\pi h[k]\delta(k-n)$$

All the terms in the sum are zero unless $n=k$, and we have

$$h[n] = \sum_{k=n}^{n} h[k] = h[n]$$