Window Method of FIR Filter Design

Assume linear phase, M even

$$H(e^{j\omega}) = \sum_{k=0}^{M} h[k]e^{-jk\omega} = e^{-j\omega M/2} A(\omega)$$

real

Consider minimizing mean squared error

$$\min_{h[k]} \frac{1}{2\pi}\int_{-\pi}^{\pi} |H_d(e^{j\omega}) - H(e^{j\omega})|^2 d\omega$$

$H_d(e^{j\omega})$ desired response

Let $H_d(e^{j\omega}) = e^{-j\omega M/2} A_d(\omega)$

$$\min_{a[k]} \frac{1}{2\pi}\int_{-\pi}^{\pi} |A_d(\omega) - A(\omega)|^2 d\omega \quad ; \quad h[k+M/2] = a[k] \xleftrightarrow{DTFT} A(\omega)$$

Parsevals Theorem

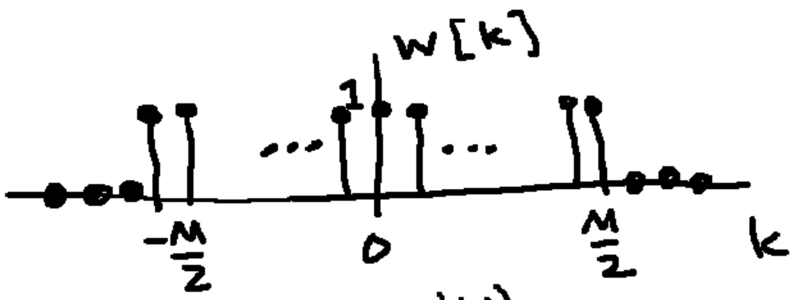
$$\min_{a[k]} \sum_{k=-\infty}^{\infty} |a_d[k] - a[k]|^2 \quad ; \quad a_d[k] \xleftrightarrow{DTFT} A_d(\omega)$$

$a[k]$ nonzero, $-\frac{M}{2} \le k \le \frac{M}{2}$

$$\min_{a[k]} \sum_{k=-M/2}^{M/2} |a_d[k]-a[k]|^2 + \sum_{k=-\infty}^{-M/2-1} |a_d[k]|^2 + \sum_{k=M/2+1}^{\infty} |a_d[k]|^2$$

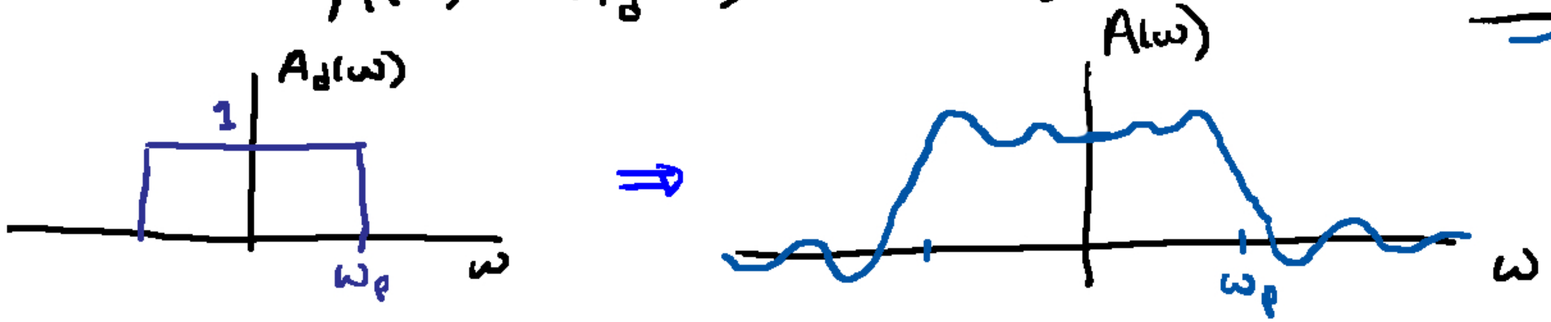
$$\Rightarrow \quad a[k] = a_d[k] \;, \quad -\frac{M}{2} \le k \le \frac{M}{2}$$

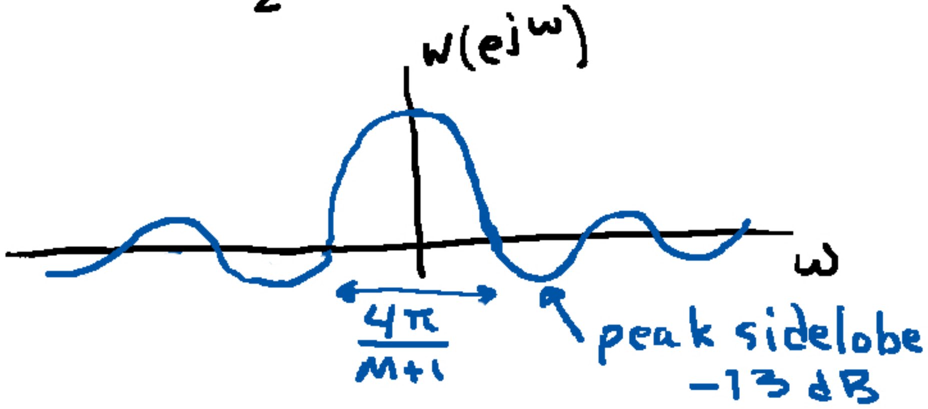
$$= a_d[k]\, w[k]$$



Nature of error $A_d(\omega) - A(\omega)$:

$$A(\omega) = A_d(\omega) * W(e^{j\omega})$$





Mainlobe width of $W(e^{j\omega}) \longrightarrow$ transition bandwidth

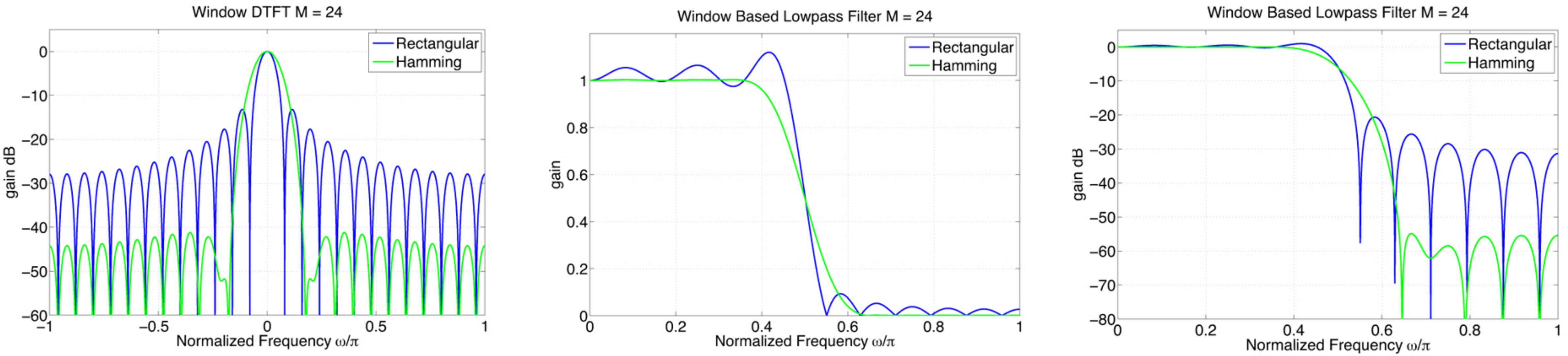
Sidelobes of $W(e^{j\omega}) \longrightarrow$ passband & stopband ripple

- -13 dB sidelobes of rectangular window $\Rightarrow$ unacceptable ripple
- Apply other windows to trade mainlobe width (transition band) for sidelobes (ripple)

Hamming window: $w[k] = \begin{cases} 0.54 - 0.46\cos\left(2\pi\left(k + \frac{M}{2}\right)\right), & -\frac{M}{2} \le k \le \frac{M}{2} \\ 0, & \text{other} \end{cases}$

Mainlobe width: $8\pi/M$

Peak sidelobe height: -41 dB



Kaiser window: approximately optimize mainlobe sidelobe tradeoff

parameters β, M :

$$w[n] = \begin{cases} \dfrac{I_0\left[\beta\left(1-\left(\frac{2n}{M}\right)^2\right)^{1/2}\right]}{I_0(\beta)}, & -\frac{M}{2} \le n \le \frac{M}{2} \\ 0, & \text{otherwise} \end{cases}$$

$I_0(\cdot)$: zeroth order modified Bessel function of 1^{ST} kind

$\beta = 0 \Rightarrow$ rectangular window

$\beta \uparrow \Rightarrow$ mainlobe width $\uparrow$, sidelobe height $\downarrow$

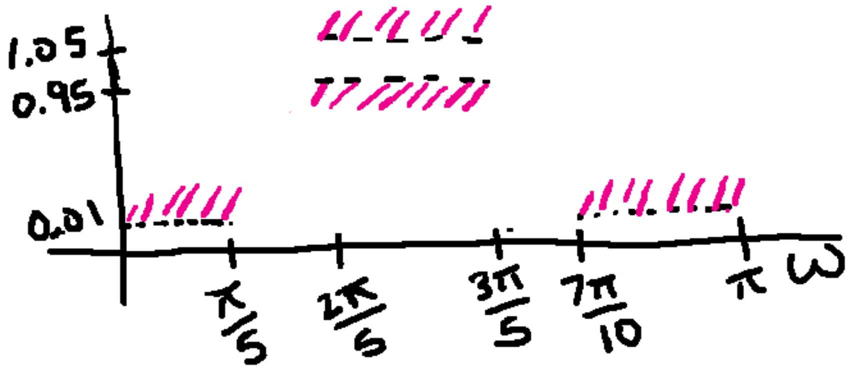
$\alpha > 0$ dB max error :

$$\beta = \begin{cases} 0.1102(\alpha - 8.7) & \alpha > 50 \\ 0.5842(\alpha-21)^{0.4} + 0.07886(\alpha-21) & 21 \le \alpha \le 50 \\ 0 & \alpha < 21 \end{cases}$$

Filter order $M = (\alpha - 8)/(2.285\Delta\omega)$

$\Delta\omega$: transition bandwidth

Example: Bandpass filter



passband error < 0.05
stopband error < 0.01
transition band $\approx \pi/10$

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F = [ pi/5,   2*pi/5,   3*pi/5,   7*pi/10]/pi;
A = [ 0   1   0 ];
Dev = [ 0.01   0.05   0.01];

[M, Wn, beta, filtype] = kaiserord(F, A, Dev);
b = fir1(M, Wn, filtype, kaiser(M+1,beta), 'noscale');
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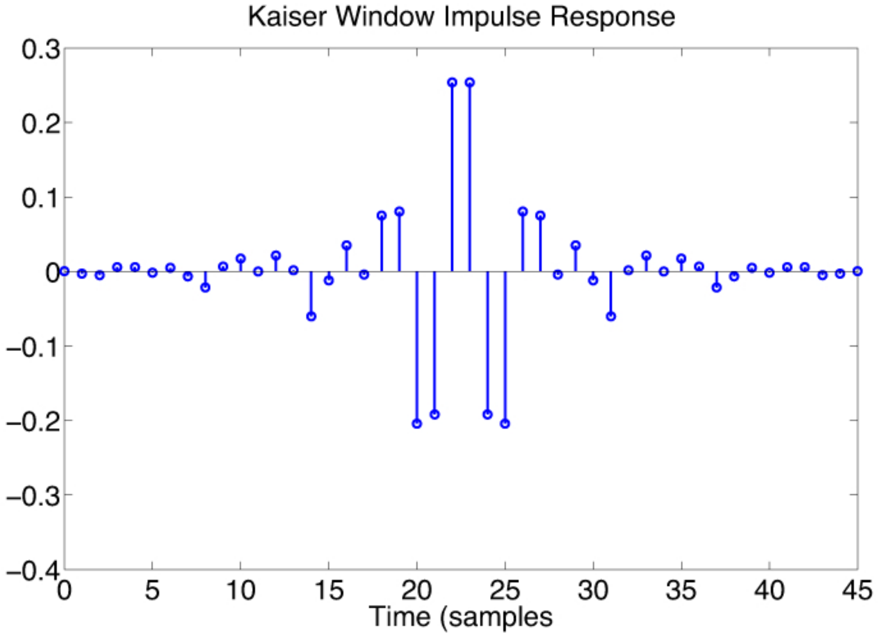
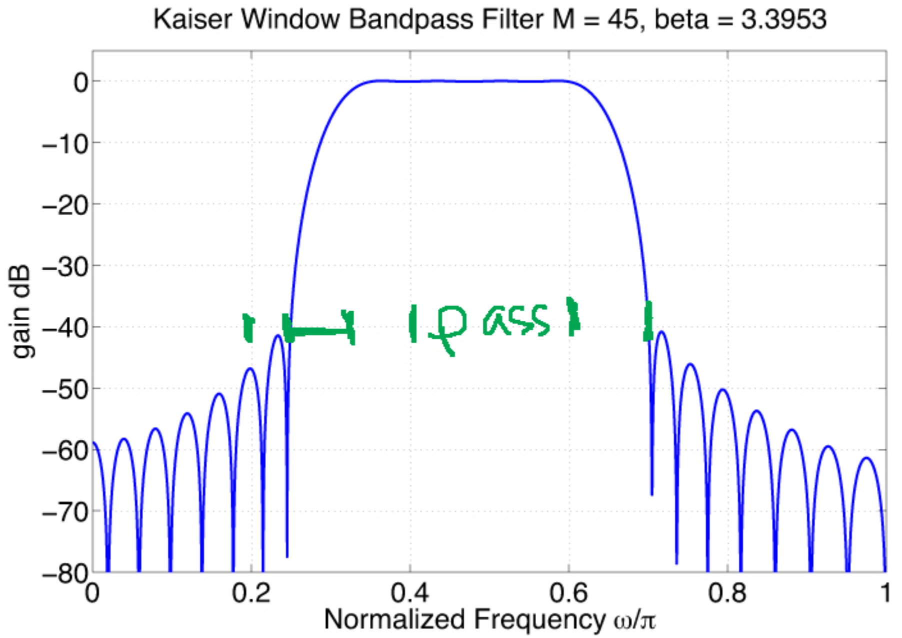
$M = 45$

$\beta = 3.3953$

Kaiser Window Bandpass Filter





passband error < 0.05
stopband error < 0.01