Minimax Optimal FIR Filter Design

Type I FIR: M even, even symmetry $h[n] = h[M-n]$ /

$$H(e^{j\omega}) = e^{-j\omega M/2} \underbrace{\sum_{k=0}^{M/2} a[k]\cos(\omega k)}_{A(\omega)}, \qquad \begin{aligned} a[0] &= h[M/2] \\ a[k] &= 2h[\tfrac{M}{2}-\ell],\ \ell = 1,\dots,\tfrac{M}{2} \end{aligned}$$

Minimax error design

$$\min_{a[k]} \max_{\omega \in F} \{ \underbrace{W(\omega)\,|H_d(\omega) - A(\omega)|}_{E(\omega)} \}$$

$H_d(\omega)$: real-valued desired response

$|H_d(\omega) - A(\omega)|$: approximation error at ω

$W(\omega)$: nonnegative weighting function

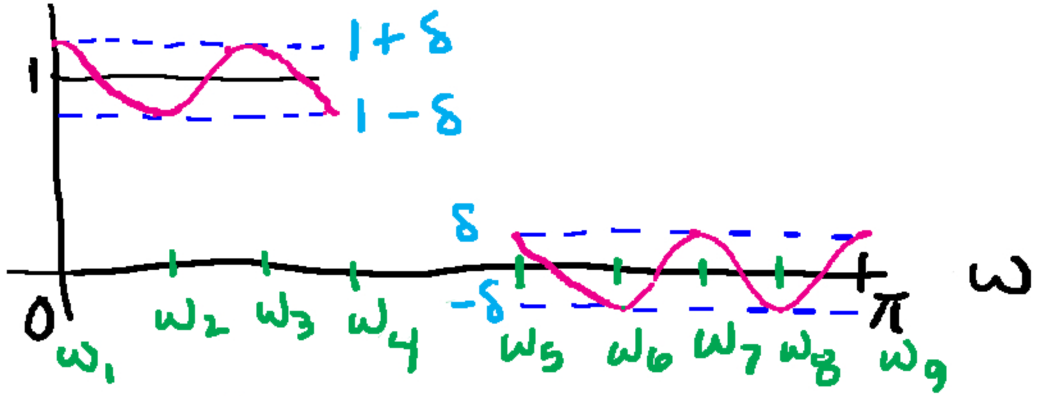
Alternation Theorem $\quad (L = M/2)$

At least $L+2$ extremal frequencies $\omega_k, k=1,2,\cdots,L+2$

1) error alternates between two equal maxima and minima: $\quad \mathcal{E}(\omega_k) = -\mathcal{E}(\omega_{k+1}) \quad , k=1,2,\ldots L+1$

2) error at ω_k equals maximum absolute error

$$\delta = |\mathcal{E}(\omega_k)| = \max_{\omega \in F} |\mathcal{E}(\omega)|$$



Design approach: find extremal frequencies ω_k, choose $a[k]$ to minimize error at ω_k

Parks McClellan Algorithm

1) Guess at initial extremal frequencies $\omega_1, \omega_2, \ldots, \omega_{L+2}$

2) Find $a[k]$, δ to satisfy alternation criterion

$$(-1)^k \delta = W(\omega_k)\left[H_d(\omega_k) - A(\omega_k)\right] \Rightarrow A(\omega_k) + \frac{(-1)^k \delta}{W(\omega_k)} = H_d(\omega_k)$$

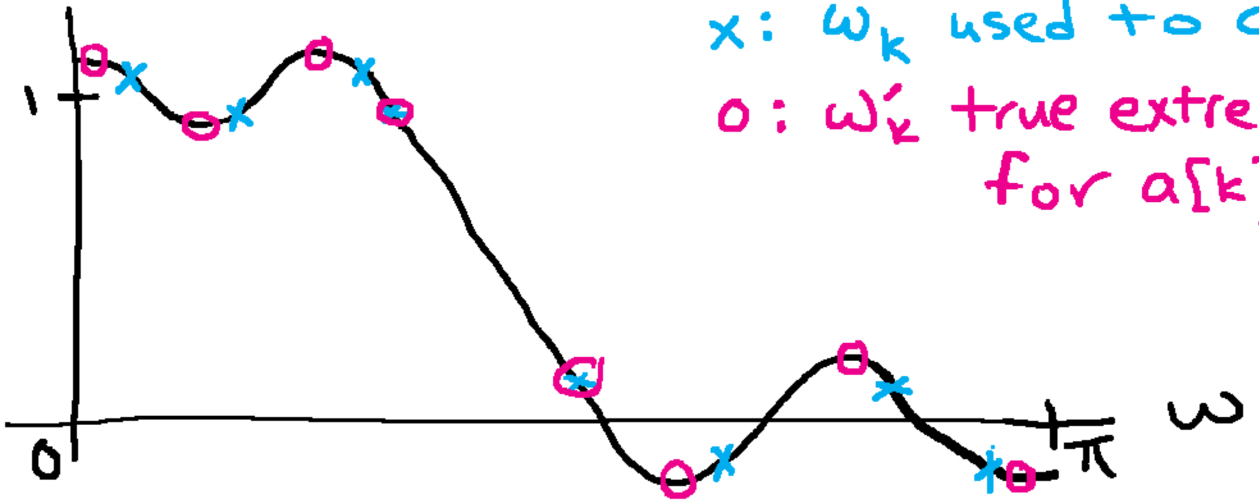
$$\begin{bmatrix} 1 & \cos\omega_1 & \cos 2\omega_1 & \cdots & \cos(L\omega_1) & \frac{-1}{W(\omega_1)} \\ 1 & \cos\omega_2 & \cos 2\omega_2 & \cdots & \cos(L\omega_2) & 1/W(\omega_2) \\ & \vdots & & & & \\ 1 & \cos\omega_{L+2} & \cdots & & \cos(L\omega_{L+2}) & \frac{(-1)^{L+2}}{W(\omega_{L+2})} \end{bmatrix} \begin{bmatrix} a[0] \\ a[1] \\ \vdots \\ a[L] \\ \delta \end{bmatrix} = \begin{bmatrix} H_d(\omega_1) \\ H_d(\omega_2) \\ \vdots \\ H_d(\omega_{L+2}) \end{bmatrix}$$

Parks McClellan Algorithm

1) Guess at initial extremal frequencies $\omega_1, \omega_2, \ldots, \omega_{L+2}$

2) Find $a[k]$, δ to satisfy alternation criterion

3) Compute true alternation frequencies ω_k' for $a[k]$



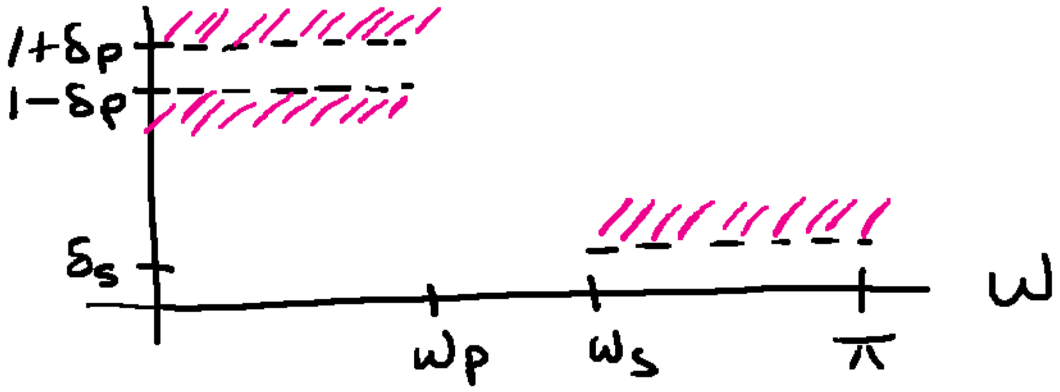
x: ω_k used to compute $a[k]$

o: ω_k' true extremal frequencies for $a[k]$

4) If $\omega_k' \neq \omega_k$, set $\omega_k = \omega_k'$ and go back to step 2.

Choosing $W(\omega)$

$$W(\omega) = \begin{cases} 1 & 0 \le \omega \le \omega_p \\ \delta_p/\delta_s & \omega_s \le \omega \le \pi \end{cases}$$



Larger weight on bands with tighter tolerance

Choosing M

(lowpass) $$M \approx \frac{-10\log_{10}(\delta_p\delta_s) - 13}{2.324\,(\omega_s - \omega_p)}$$