Frequency Sampling FIR Filter Design

FIR filter

$$y[n] = \sum_{k=0}^{M} b_k x[n-k]$$

$$H(z) = \sum_{k=0}^{M} b_k z^{-k} \implies H(e^{j\omega}) = \sum_{k=0}^{M} b_k e^{-jk\omega}$$

$$h[n] = \begin{cases} b_n & 0 \le n \le M \\ 0 & \text{otherwise} \end{cases}$$

Frequency sampling design: find b_k so $H(e^{j\omega_\ell}) \approx A_\ell e^{j\phi_\ell}$ for $1 \le \ell \le N$

Write $H(e^{j\omega})$ as an inner product of vectors

$$H(e^{j\omega}) = \sum_{k=0}^{M} b_k e^{-jk\omega} = \underbrace{[1 \;\; e^{-j\omega} \;\; e^{-j2\omega} \; \cdots \; e^{-jM\omega}]}_{\underline{d}(\omega)} \underbrace{\begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ \vdots \\ b_M \end{bmatrix}}_{\underline{b}}$$

$$= \underline{d}(\omega)\,\underline{b}$$

Require conjugate symmetry for real $\underline{b}$

If $H(e^{j\omega_\ell}) = A_\ell e^{j\phi_\ell}$, then $H(e^{-j\omega_\ell}) = A_\ell e^{-j\phi_\ell}$

$$\begin{bmatrix} \underline{d}(\omega_\ell) \\ \underline{d}(-\omega_\ell) \end{bmatrix} \underline{b} = \begin{bmatrix} A_\ell e^{j\phi_\ell} \\ A_\ell e^{-j\phi_\ell} \end{bmatrix}$$

2 equations for each ℓ

$M+1$ unknowns in $\underline{b}$

Suppose $N/2$ constraint frequencies

$$\underbrace{\begin{bmatrix} \underline{d}(\omega_1) \\ \underline{d}(-\omega_1) \\ \underline{d}(\omega_2) \\ \underline{d}(-\omega_2) \\ \vdots \\ \underline{d}(\omega_{N/2}) \\ \underline{d}(-\omega_{N/2}) \end{bmatrix}}_{\text{full rank for distinct } \omega_\ell} \underline{b} = \begin{bmatrix} A_1 e^{j\phi_1} \\ A_1 e^{-j\phi_1} \\ A_2 e^{j\phi_2} \\ A_2 e^{-j\phi_2} \\ \vdots \\ A_{N/2} e^{j\phi_{N/2}} \\ A_{N/2} e^{-j\phi_{N/2}} \end{bmatrix}$$

N equations

$M+1$ unknowns

$$\underset{N \times M+1}{\underline{D}}\, \underline{b} = \underline{f}$$

If $N \leq M+1$, we can exactly satisfy the design constraints

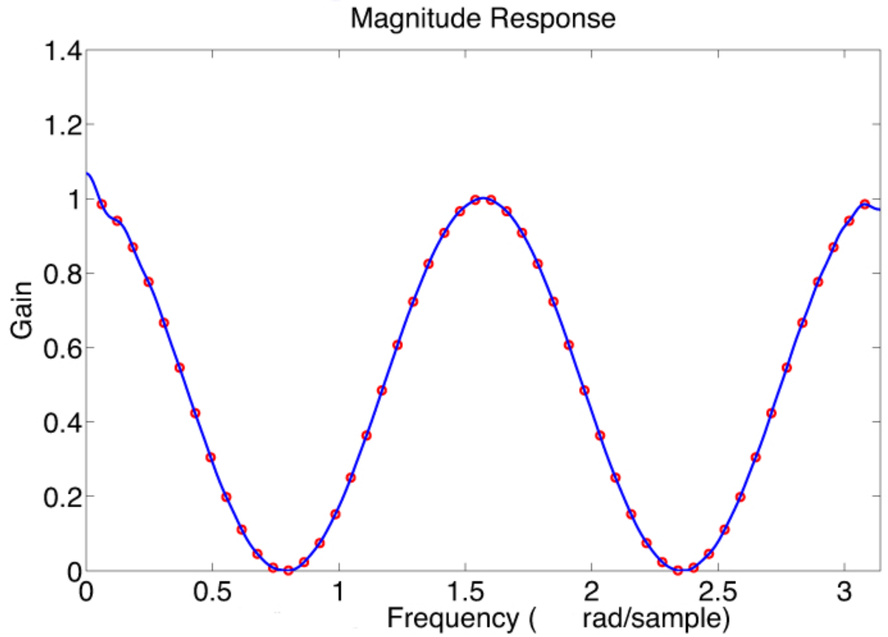
- Exact control of frequency response at up to $(M+1)/2$ frequencies
- No control of response at unspecified frequencies
  - Analogous to fitting polynomials $H(e^{j\omega}) = \sum_{k=0}^{M} b_k e^{-jk\omega}$
  - Specify gradual changes in $A_\ell e^{j\phi_\ell}$

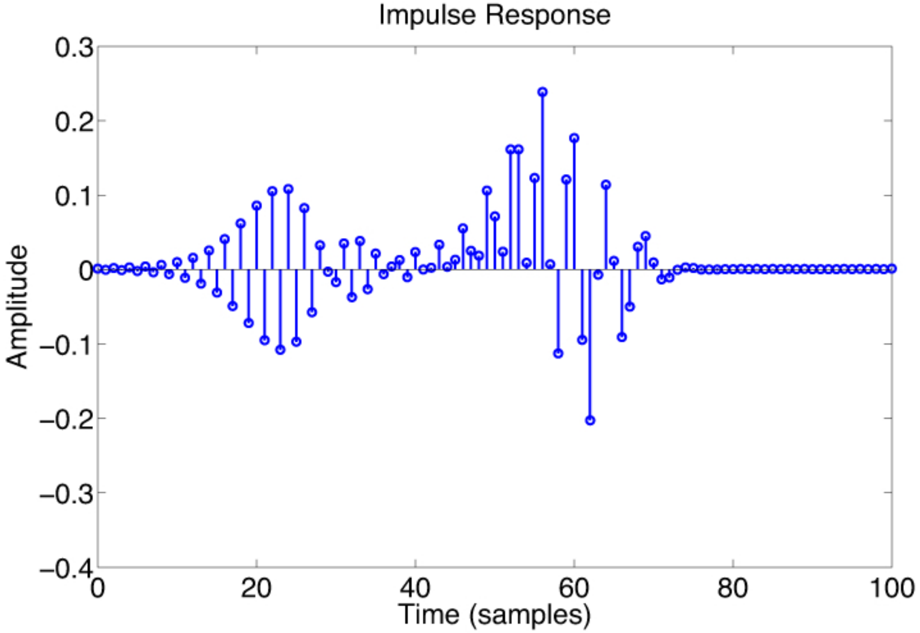
Example: $\omega_\ell = \frac{\ell\pi}{51}$, $\ell = 1, 2 \cdots, 50$

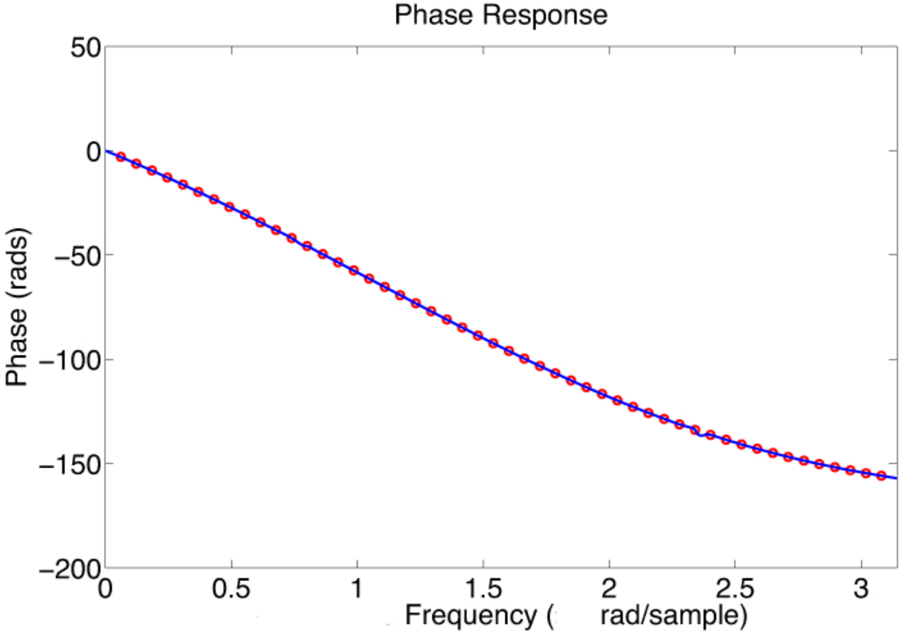
$$A_\ell = (1 + \cos(4\omega_\ell))/2$$

$$\phi_\ell = -50\,\omega_\ell(1 + 0.2\sin(\omega_\ell))$$

$$M = 100$$





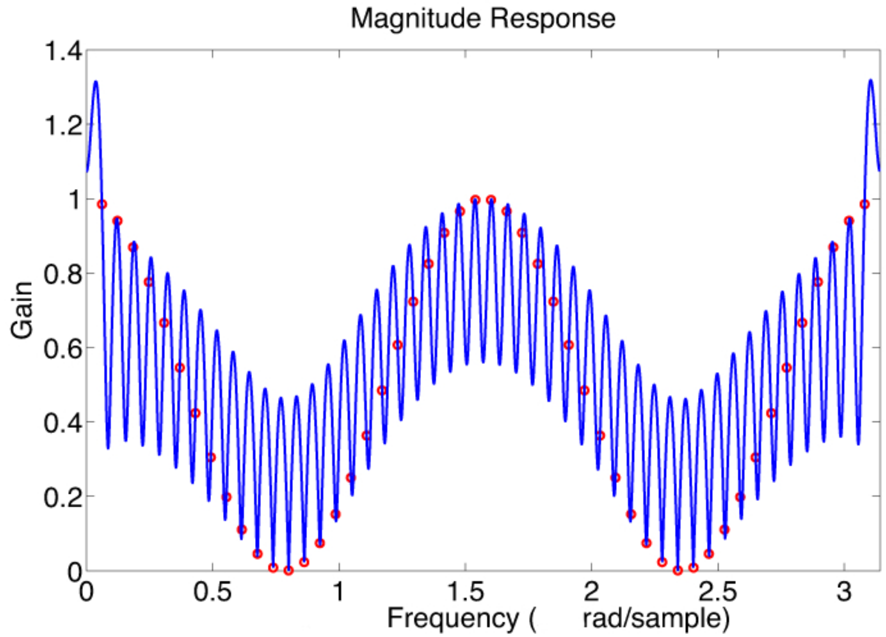


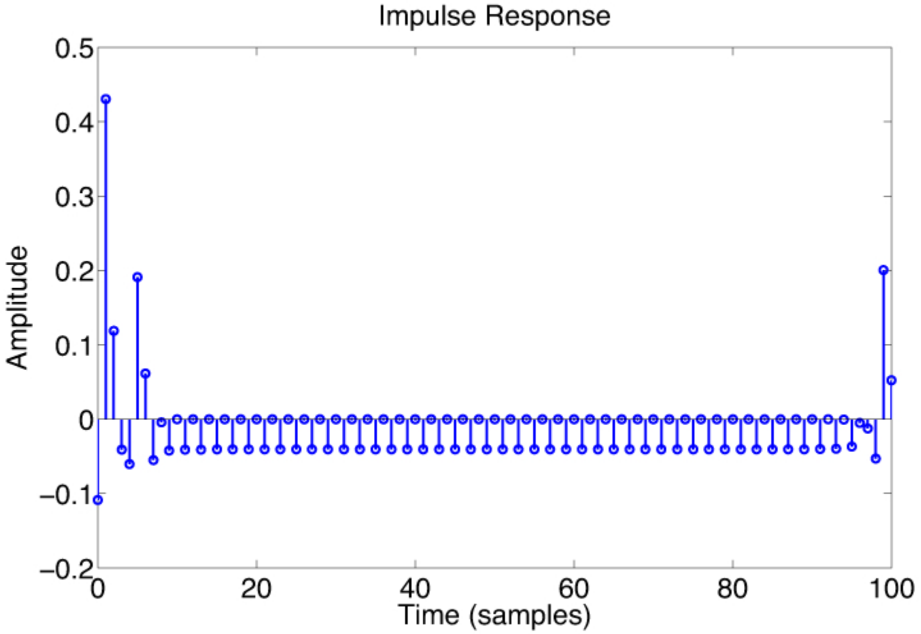
Example: $\omega_\ell = \dfrac{\ell\pi}{51}$, $\ell = 1, 2, \cdots, 50$

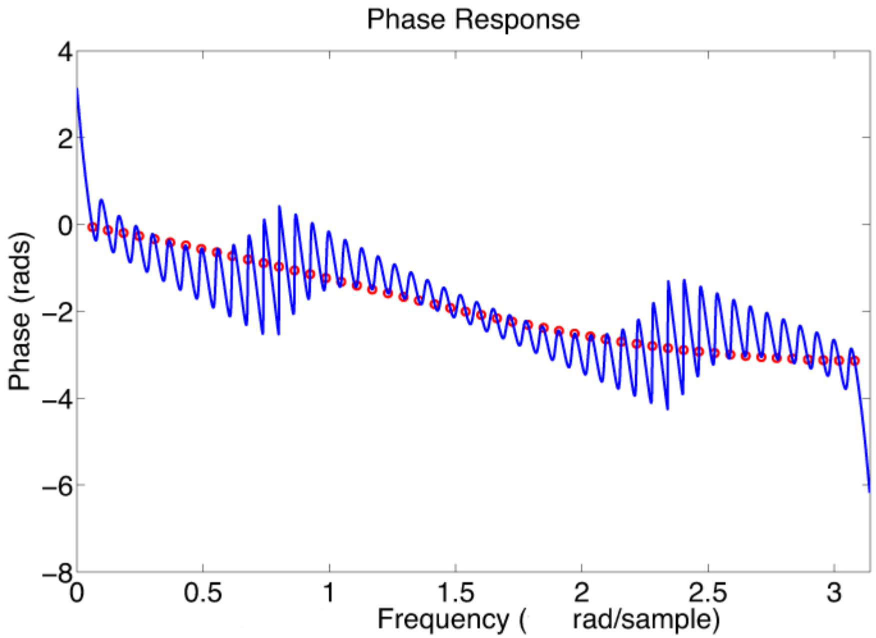
$$A_\ell = (1 + \cos(4\omega_\ell))/2$$

$$\phi_\ell = -\omega_\ell(1 + 0.3\sin(\omega_\ell))$$

$$M = 100$$







Least Squares Solutions: $N > M+1$

$\underline{D} \cdot \underline{b} = \underline{f}$ more equations than unknowns

$N \times (M+1) \cdot (M+1) \times 1 = N \times 1$

$\min_{\underline{b}} |\underline{D}\underline{b} - \underline{f}|^2 \Rightarrow \underline{b} = (\underline{D}^H \underline{D})^{-1} \underline{D}^H \underline{f}$

$(\cdot)^H$: complex conjugate transpose

- Don't obtain exact solution: $H(e^{j\omega_\ell}) \neq A_\ell e^{j\phi_\ell}$
- As $N \uparrow$, criterion approaches

$$\min_{\underline{b}} \int_{\omega=\pi} |H(e^{j\omega}) - A(\omega) e^{j\phi(\omega)}|^2 d\omega$$