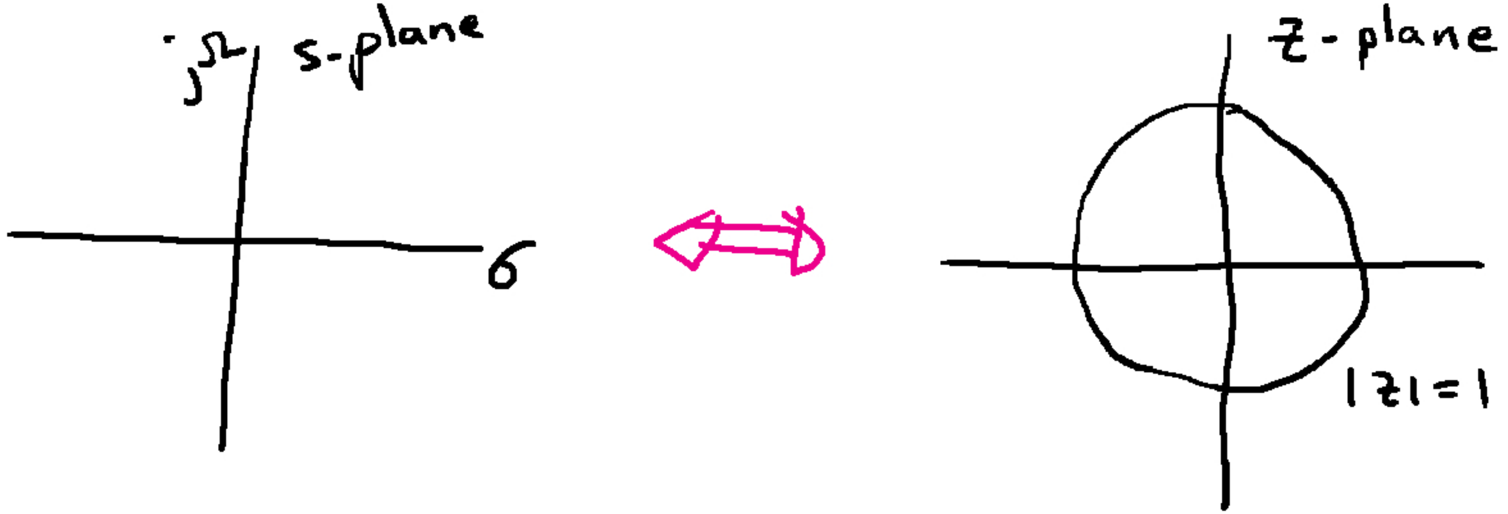
The Bilinear Transform

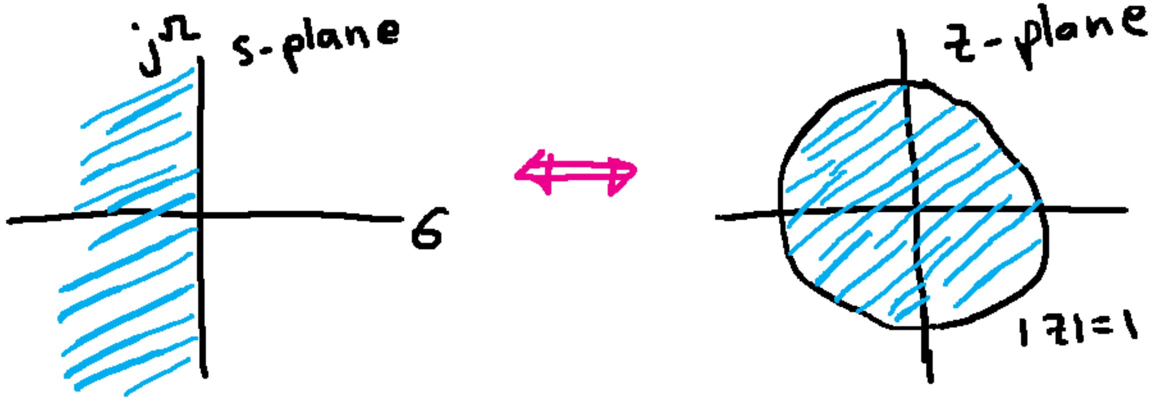
Map continuous-time systems to discrete-time systems by relating s and z

$$s = 2\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right)$$ bilinear transform

$$z = \frac{1 + s/2}{1 - s/2}$$ inverse bilinear transform



Property 1: left half of s-plane ($\sigma < 0$) maps to interior of unit circle in the z plane



$$z = \frac{1 + s/2}{1 - s/2} = \frac{1 + \sigma/2 + j\Omega/2}{1 - \sigma/2 - j\Omega/2}$$

if $\sigma < 0$, $|z| < 1$

$\sigma > 0$, $|z| > 1$

Stable CT systems $\Longleftrightarrow$ stable DT systems

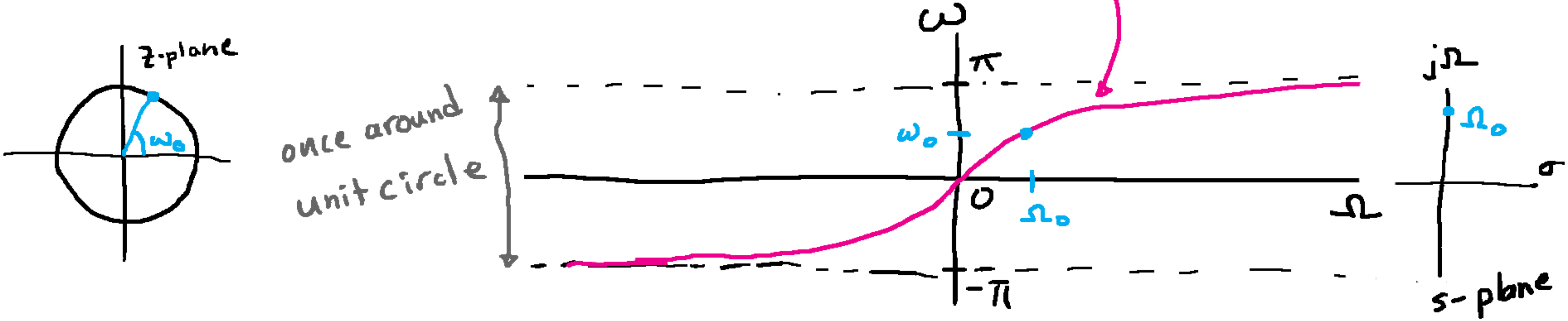
Property 2: $j\Omega$ axis maps to unit circle via a nonlinear warping

$$z = \frac{1 + s/2}{1 - s/2} \xrightarrow[s = j\Omega]{} z = \frac{1 + j\Omega/2}{1 - j\Omega/2} \underset{\left|\frac{a}{a^*}\right| = 1}{\Longrightarrow} e^{j\omega} = \frac{1 + j\Omega/2}{1 - j\Omega/2}$$

solve for Ω

$$j\Omega = 2\,\frac{1 - e^{-j\omega}}{1 + e^{-j\omega}} = 2\,\frac{2je^{-j\omega/2}\sin(\omega/2)}{2e^{-j\omega/2}\cos(\omega/2)}$$

$$\Omega = 2\tan(\omega/2) \quad \text{or} \quad \omega = 2\tan^{-1}(\Omega/2)$$

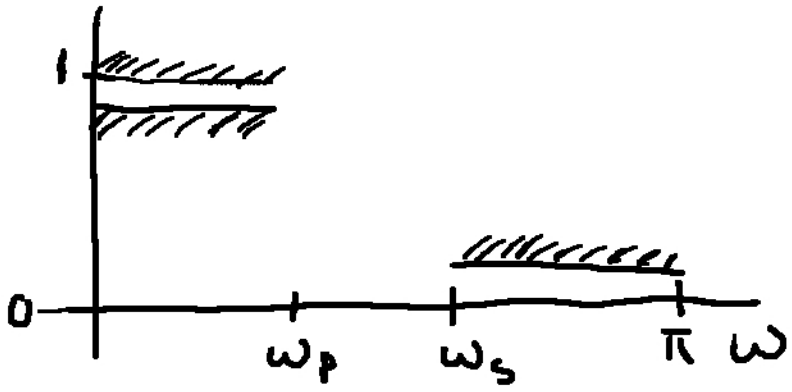


Use: Design $H_c(s)$, then $H(z) = H_c(s)\Big|_{s = 2\frac{1-z^{-1}}{1+z^{-1}}}$

Compensate for warping between Ω and ω when designing $H_c(s)$!

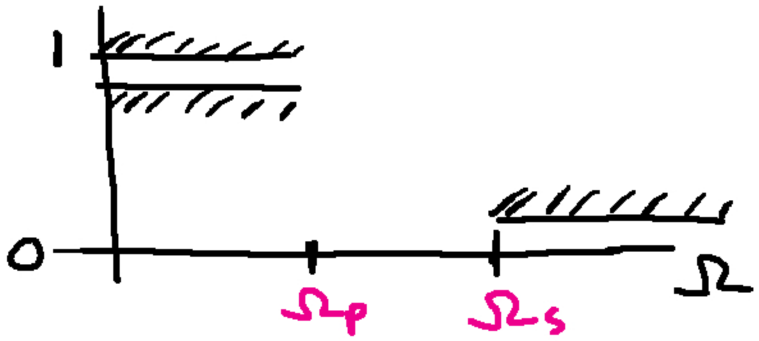
Prewarp critical (stopband, passband) frequencies

Example:



DT Filter Specifications

$$\rightarrow \quad \Omega_p = 2\tan\left(\frac{\omega_p}{2}\right)$$
$$\Omega_s = 2\tan\left(\frac{\omega_s}{2}\right)$$



Required CT filter Specifications

Example: Design 1st order discrete-time lowpass Butterworth filter with 1/2 power frequency $\omega = \pi/4$ rads

1) Prewarp. Continuous-time filter 1/2 power frequency

$$\Omega_c = 2\tan\left(\frac{\pi/4}{2}\right) = 2\tan(\pi/8) \approx 0.828 \text{ rads/sec}$$

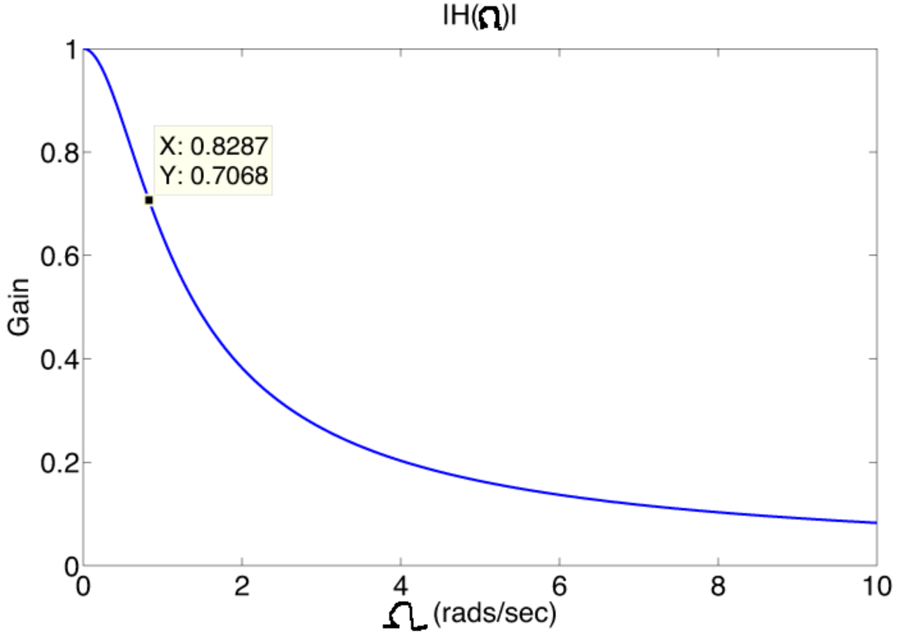
2) CT filter design.

$$H_{LP}(s) = \frac{1}{s+1}$$ is Butterworth with 1/2 power freq 1 rads/sec

LP to LP frequency transformation

$$s = \tilde{s}/0.828$$

$$H_c(\tilde{s}) = \frac{0.828}{\tilde{s} + 0.828}$$



3) Bilinear transform.

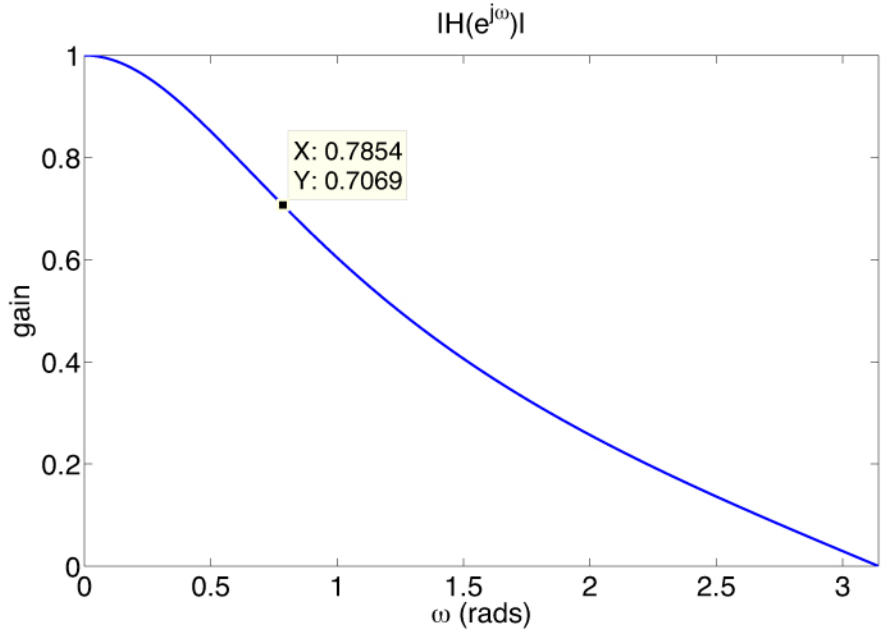
$$H(z) = H_c(\tilde{s})\Big|_{\tilde{s}=2\frac{1-z^{-1}}{1+z^{-1}}}$$

$$= \frac{0.828}{2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+0.828} = \frac{0.414(1+z^{-1})}{1-z^{-1}+0.414+0.414z^{-1}}$$

$$= \frac{0.414(1+z^{-1})}{1.414-0.586z^{-1}}$$

$$= \frac{0.414}{1.414}\left(\frac{1+z^{-1}}{1-\frac{0.586}{1.414}z^{-1}}\right)$$

$\omega = \pi/4 \approx .7854$



Remapping of poles/zeros

$$s_k \longrightarrow z_k = \frac{1 + s_k/2}{1 - s_k/2}$$

$$s_k = \infty \longrightarrow z_k = -1$$

- Requires gain normalization