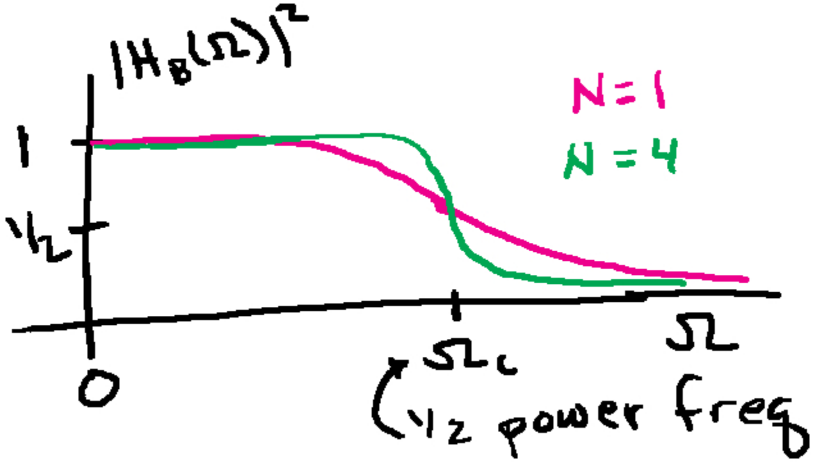
Continuous Time Butterworth Filters

Magnitude response squared

$$|H_B(\Omega)|^2 = \frac{1}{1+(\Omega/\Omega_c)^{2N}}$$



$N \uparrow \Rightarrow$ sharper cutoff

Monotone decreasing

$|H_B(\Omega)|^2 \downarrow$ as $\Omega \uparrow$

Maximally flat

For $\Omega/\Omega_c < 1$

$$|H_B(\Omega)|^2 = 1 - \left(\frac{\Omega}{\Omega_c}\right)^{2N} + \left(\frac{\Omega}{\Omega_c}\right)^{4N} - \cdots$$

1^{st} $2N-1$ derivatives zero $\Omega = 0$

In s-plane

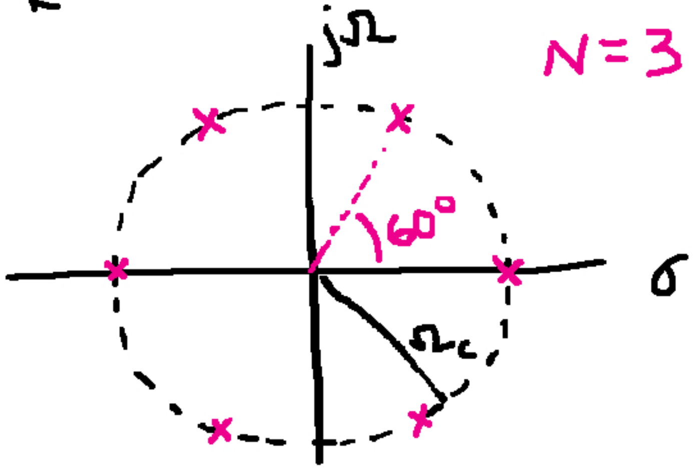
$$|H_B(\Omega)|^2 = H_B(\Omega)\,H_B^*(\Omega) = H_B(\Omega)\,H_B(-\Omega) \Rightarrow$$

$$H_B(s)\,H_B(-s) = \frac{1}{1+\left(\frac{s}{j\Omega_c}\right)^{2N}} \qquad (\text{use } s = j\Omega)$$

Poles of $H_B(s)H_B(-s)$

$$\left(\frac{s_k}{j\Omega_c}\right)^{2N} = -1 \Rightarrow s_k = \Omega_c\, e^{j\frac{\pi}{2N}(2k+N-1)} \qquad k=0,1,\ldots 2N-1$$

N=3

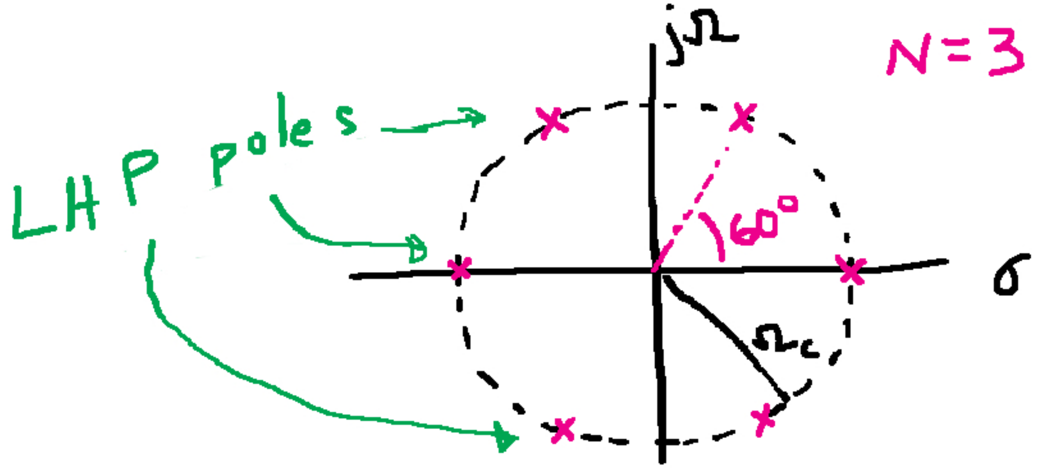


2N poles spaced by $\frac{360°}{2N}$ at radius Ω_c

For stable/causal $H_B(s)$, choose poles in the left half plane

If $N = 3$

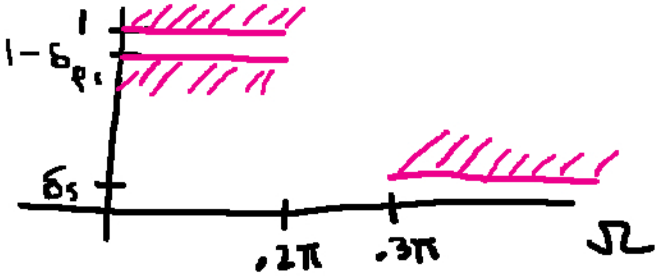
$$s_1 = -\Omega_c$$
$$s_2 = \Omega_c e^{j2\pi/3}$$
$$s_3 = \Omega_c e^{-j2\pi/3}$$



$$H_B(s) = \frac{\Omega_c^3}{(s+\Omega_c)(s-\Omega_c e^{j2\pi/3})(s-\Omega_c e^{-j2\pi/3})}$$

$$= \frac{\Omega_c^3}{(s+\Omega_c)(s^2+\Omega_c s + \Omega_c^2)}$$

Design Example: choose Ω_c, N



$$1 \geq |H_B(\Omega)| \geq 0.89 = 1-\delta_{p_1}, \quad 0 \leq |\Omega| < 0.2\pi$$

$$|H_B(\Omega)| \leq 0.175 = \delta_s, \quad |\Omega| \geq 0.3\pi$$

Response is monotone $\Rightarrow$ use bandedges

passband $\quad |H_B(\pi/5)|^2 = \dfrac{1}{1+\left(\frac{\pi}{5\Omega_c}\right)^{2N}} \geq (0.89)^2$

stopband $\quad |H_B(\frac{3\pi}{10})|^2 = \dfrac{1}{1+\left(\frac{3\pi}{10\Omega_c}\right)^{2N}} \leq (0.175)^2$

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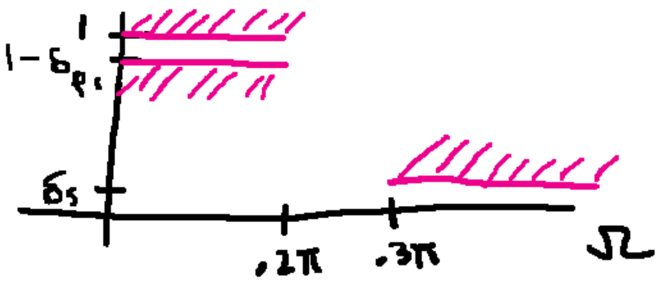

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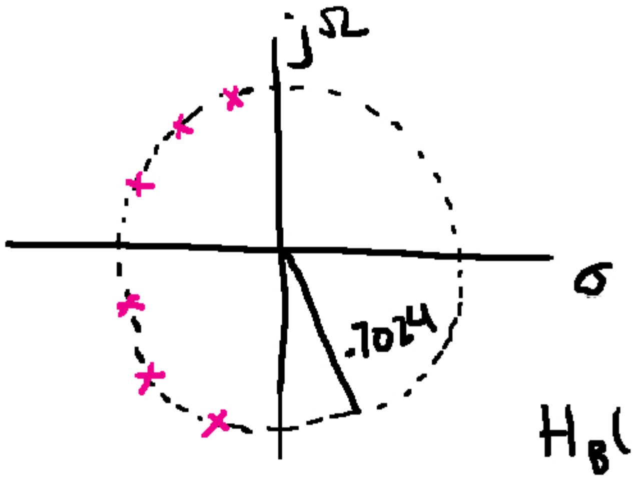
Force equality at passband edge to obtain

$$N = 6, \quad \Omega_c = 0.7024$$

Pole locations



$$s_{1,2} = 0.7024\, e^{\pm j 7\pi/12}$$

$$s_{3,4} = 0.7024\, e^{\pm j 3\pi/4}$$

$$s_{5,6} = 0.7074\, e^{\pm j 11\pi/12}$$

$$H_B(s) = \frac{0.7074^6}{(s^2 + 0.3636 s + 0.7074^2)(s^2 + 0.9934 s + 0.7074^2)(s^2 + 1.357 s + 0.7074^2)}$$

Satisfies specifications