Impulse Response and Poles and Zeros

How do the poles/zeros of an LTI system determine / impulse response? transient response?

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k] \Longrightarrow H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

$$h[n] \overset{z}{\longleftrightarrow} H(z)$$

Partial Fraction Expansion of H(z)

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{a_0 \prod_{k=1}^{N} (1 - d_k z^{-1})} = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1}^{N_1} \frac{A_k}{1 - d_k z^{-1}} + \sum_{k=N_1}^{N_2} \frac{C_k d_k z^{-1}}{(1 - d_k z^{-1})^2} + \dots$$

$$h[n] = \sum_{r=0}^{M-N} B_r \delta[n-r] + \sum_{k=1}^{N_1} A_k (d_k)^n u[n] + \sum_{n=N_1}^{N_2} C_k n (d_k)^n u[n] + \dots$$

r^{th} order pole at zero $\longrightarrow$ $B_r \delta[n-r]$

Single pole at d_k $\longrightarrow$ $A_k (d_k)^n u[n]$

Double pole at d_k $\longrightarrow$ $C_k \, n \, (d_k)^n u[n]$

Example:

$$H(z) = 2z^{-2} + \frac{1}{1-\frac{3}{4}z^{-1}} + \frac{\frac{1}{2}(-\frac{1}{2})z^{-1}}{1+\frac{1}{2}z^{-1}}$$

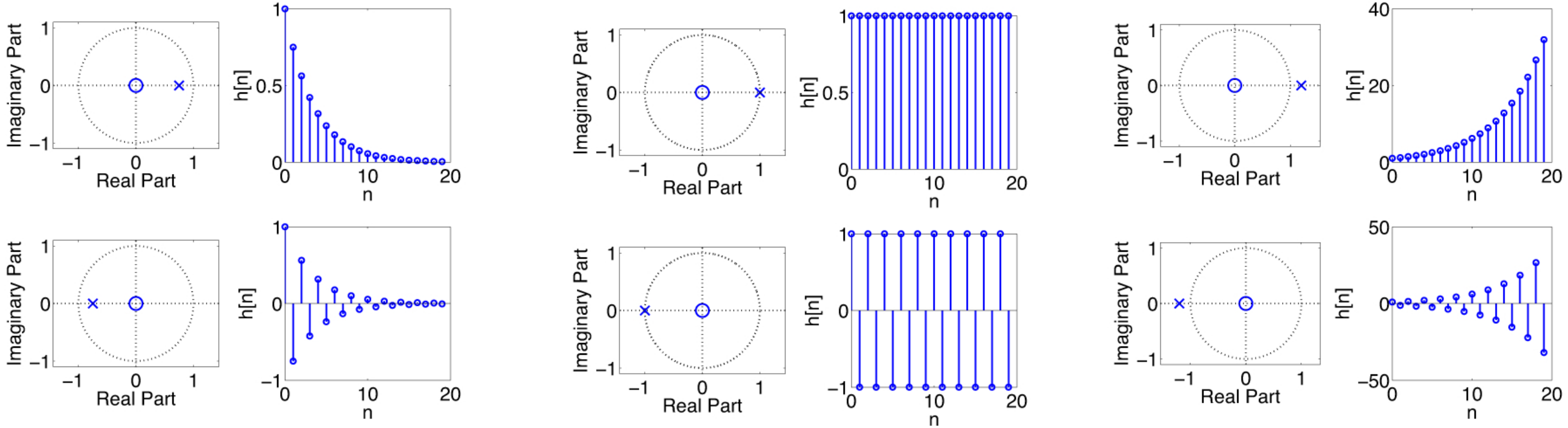
$$h[n] = 2\delta[n-2] + \left(\tfrac{3}{4}\right)^n u[n] + \tfrac{1}{2} n \left(-\tfrac{1}{2}\right)^n u[n]$$

Behavior of impulse response contribution depends on pole location and multiplicity

1) Single real pole

$$\frac{1}{1 - a z^{-1}} \longleftrightarrow a^n u[n]$$

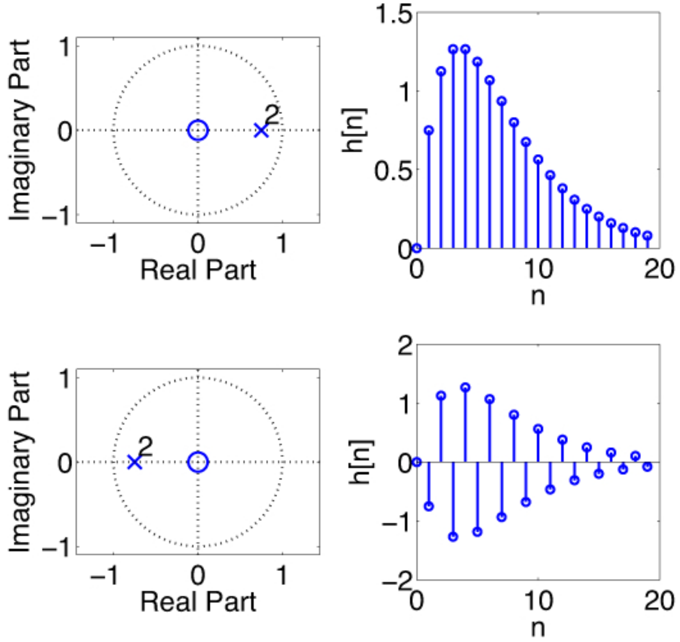
$|a| < 1$ $\qquad$ $|a| = 1$ $\qquad$ $|a| > 1$



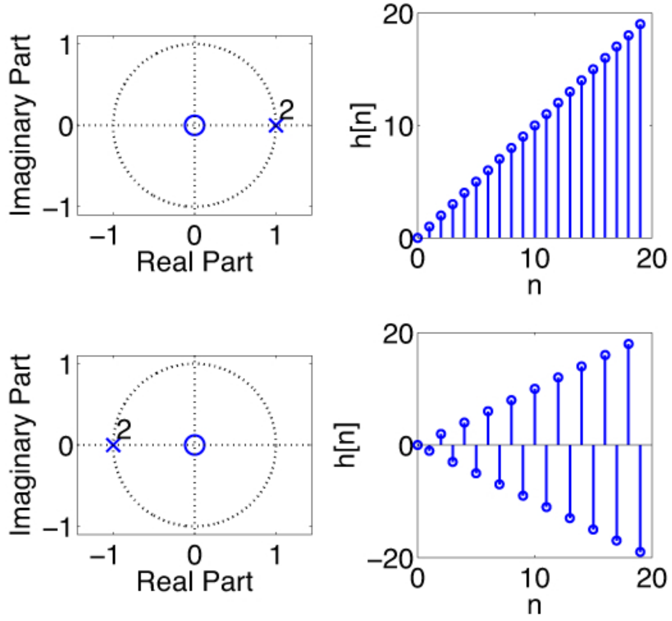
2) 2^{nd} order real pole

$$\frac{a z^{-1}}{(1 - a z^{-1})^2} \longleftrightarrow n a^n u[n]$$

$|a| < 1$



$|a| = 1$



3) Complex conjugate pair of poles

– poles at $re^{\pm j\omega_0}$

$$\frac{1 - r\cos(\omega_0)z^{-1}}{1 - 2r\cos(\omega_0)z^{-1} + r^2 z^{-2}} \longleftrightarrow r^n \cos(\omega_0 n)\, u[n]$$

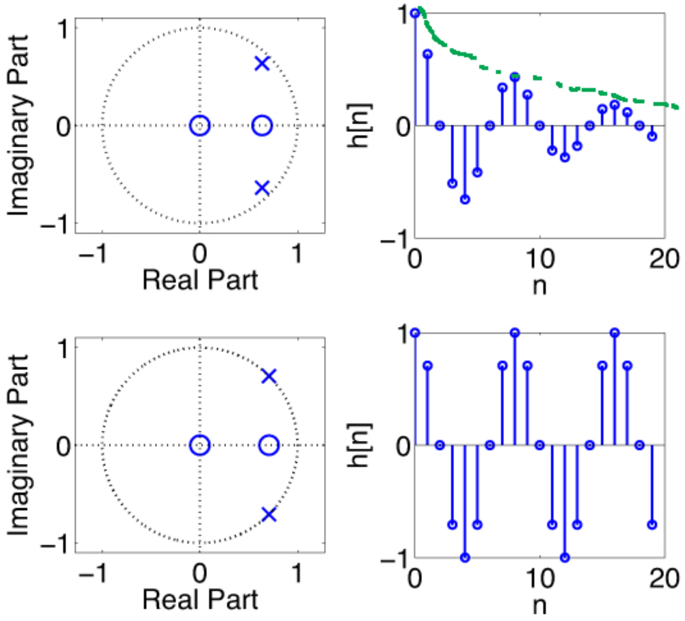
$r^n e^{j\omega_0 n}$

$r^n e^{-j\omega_0 n}$

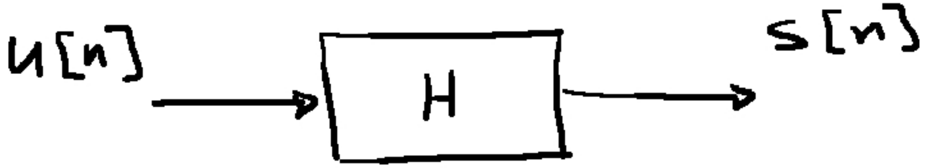
$\omega_0 = \pi/4$

$|r| < 1$

$|r| = 1$



Impulse response duration → transient response time



"step response"

$$u[n] = \sum_{k=-\infty}^{n} \delta[k] \implies s[n] = \sum_{k=-\infty}^{n} h[k]$$

poles near

$z = 0 \Rightarrow$ fast response

$|z| = 1 \Rightarrow$ slow response

step response

$r = 0.9$

$r = 0.25$