Minimum Phase and All-Pass Systems

A stable, causal system has a stable, causal inverse

if and only if

all poles and zeros are inside $|z|=1$

called: <u>Minimum phase system</u>

Can show that phase lag of a system with poles/zeros inside $|z|=1$ is less than that of any other system with identical magnitude response

Any rational system function $H(z)$

$$H(z) = \underbrace{H_{min}(z)}_{\text{minimum phase}} \underbrace{H_{ap}(z)}_{\text{all pass}}$$

no zeros on $|z|=1$

All pass : $|H_{ap}(e^{j\omega})| = 1$

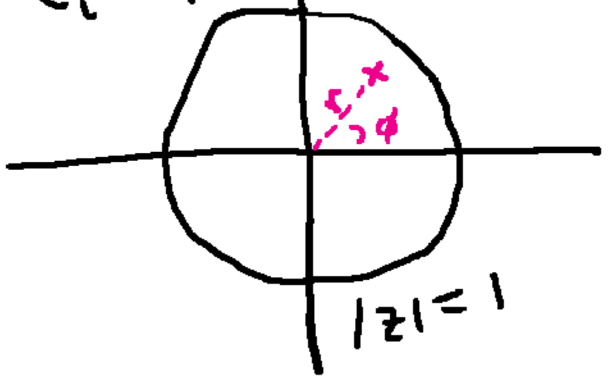
All pass $\Longleftrightarrow$ poles and zeros in conjugate reciprocal pairs

$$H_{ap}(z) = \prod_{i=1}^{P} \frac{z^{-1} - c_i^*}{1 - c_i z^{-1}}$$

poles: $c_i = re^{j\phi}$

zeros: $1/c_i^* = \frac{1}{r}e^{j\phi}$

$\circ\ \frac{1}{r}e^{j\phi}$



To show $|H_{ap}(e^{j\omega})| = 1$, consider $P = 1$ $\quad |e^{-j\omega}| = 1$ $\quad$ 3

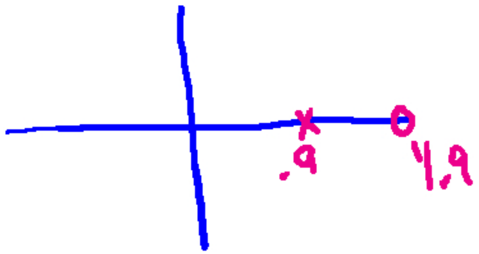
$$|H_{ap}(e^{j\omega})| = \left| \frac{e^{-j\omega} - c^*}{1 - c\,e^{-j\omega}} \right| = \left| \frac{e^{-j\omega}(1 - c^* e^{j\omega})}{1 - c\,e^{-j\omega}} \right|$$

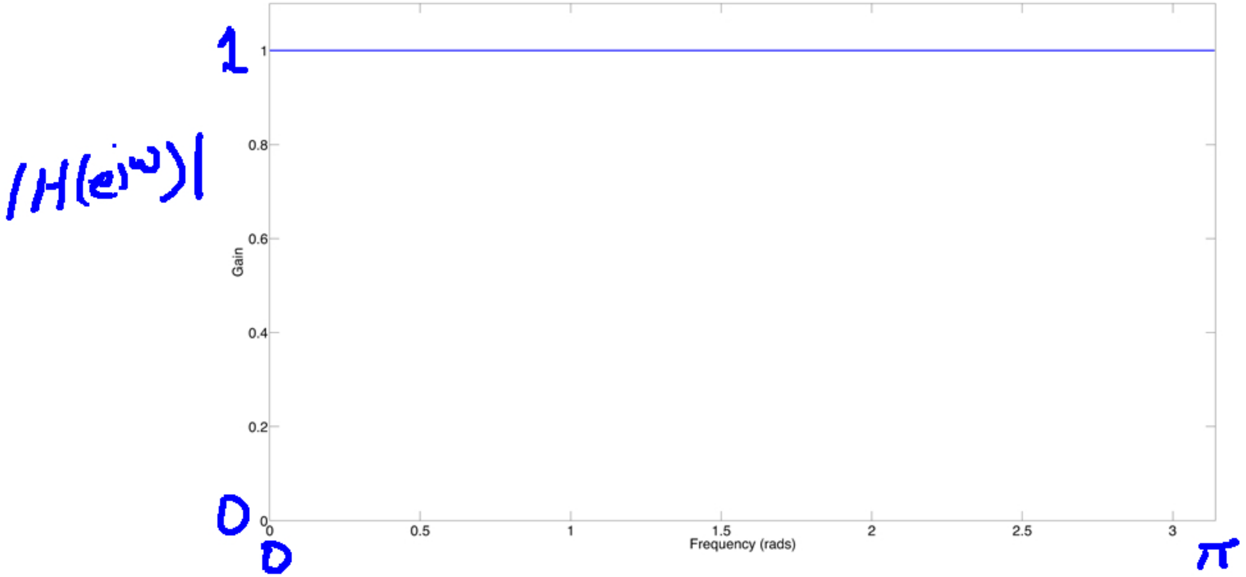
$$= \left| \frac{1 - c^* e^{j\omega}}{\underbrace{1 - c\,e^{-j\omega}}_{b}} \right| = \frac{|b^*|}{|b|} = 1$$

$$\frac{z^{-1} - c^*}{1 - c\,z^{-1}}$$

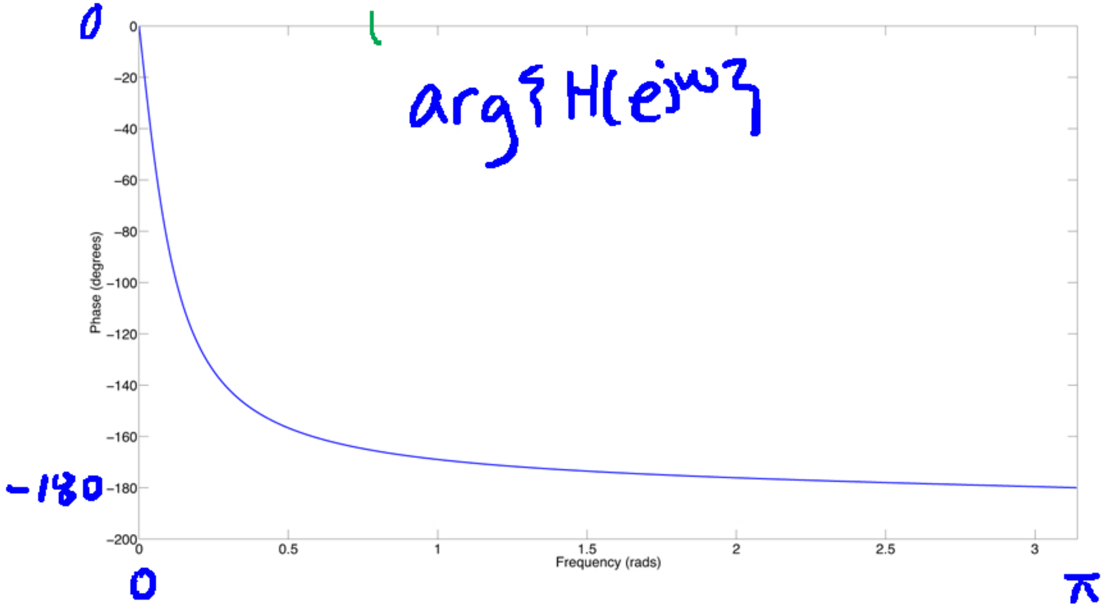
Example: All-pass System

$$H(z) = \frac{z^{-1} - c^*}{1 - c z^{-1}}$$

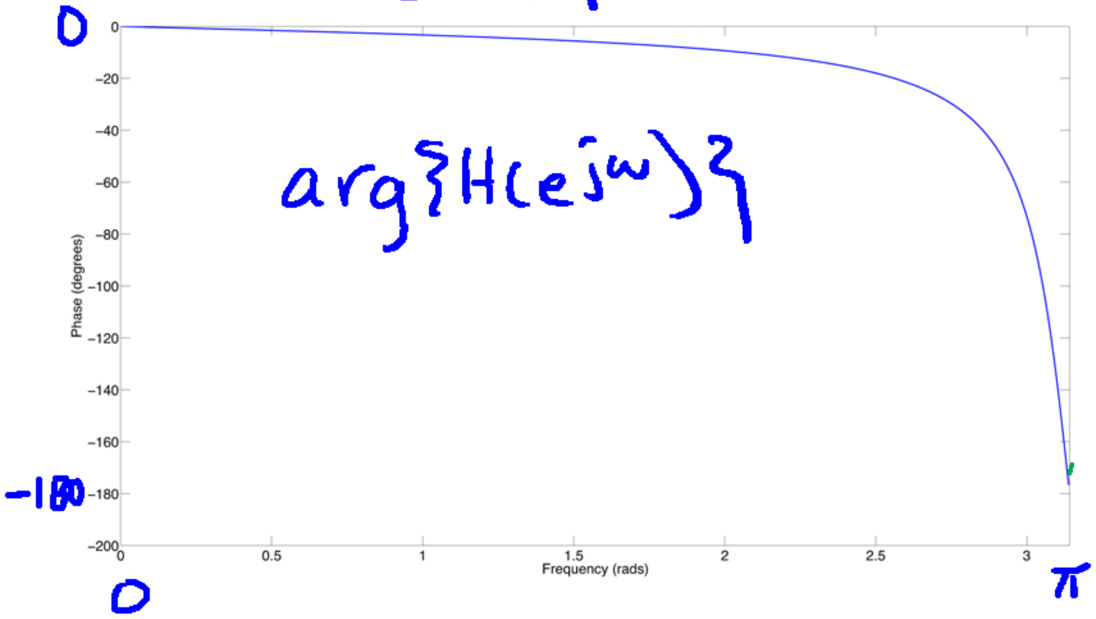




$c = 0.9$



$c = -0.9$



To factor $H(z) = H_{min}(z)\, H_{ap}(z)$:

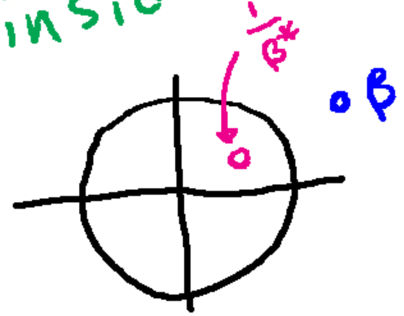
1) Take zeros that lie outside $|z|=1$ and move to $H_{ap}(z)$
2) Add poles to $H_{ap}(z)$ in conjugate reciprocal locations of zeros
3) Put zeros in $H_{min}(z)$ to cancel poles added to $H_{ap}(z)$

Suppose $H(z) = H_1(z)(1-\beta z^{-1})$; $|\beta|>1$, $H_1(z)$ min phase

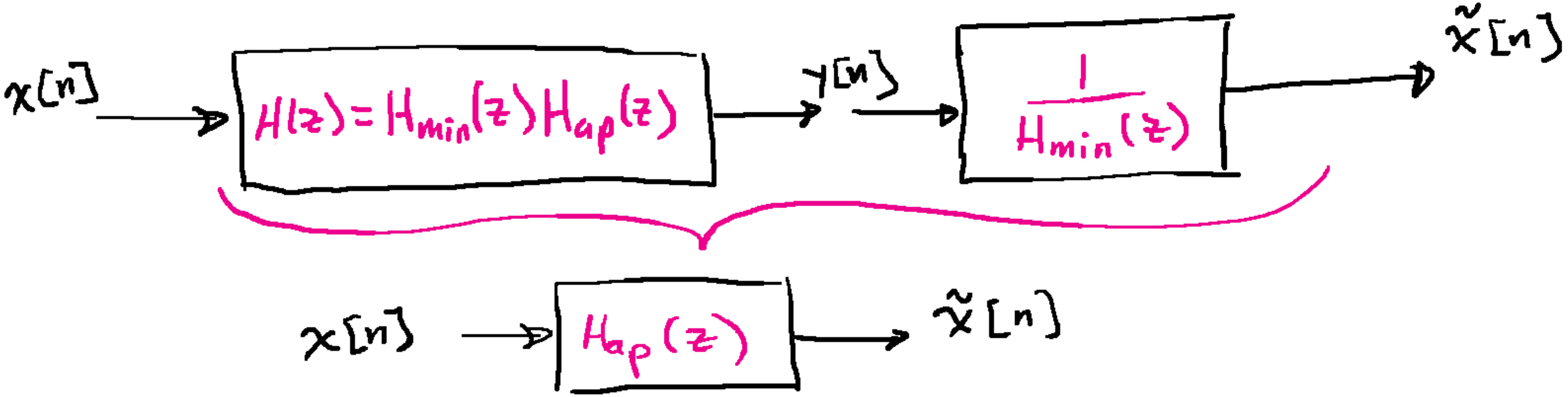
1) $H(z) = H_1(z)(-\beta)\left(z^{-1} - \frac{1}{\beta}\right)$

2+3) $H(z) = \underbrace{H_1(z)(-\beta)\left(1 - \frac{1}{\beta^*} z^{-1}\right)}_{H_{min}(z)} \; \underbrace{\frac{z^{-1} - \frac{1}{\beta}}{1 - \frac{1}{\beta^*} z^{-1}}}_{H_{ap}(z)}$

zero at $z=\beta$ in $H(z)$ reflected inside $|z|=1$

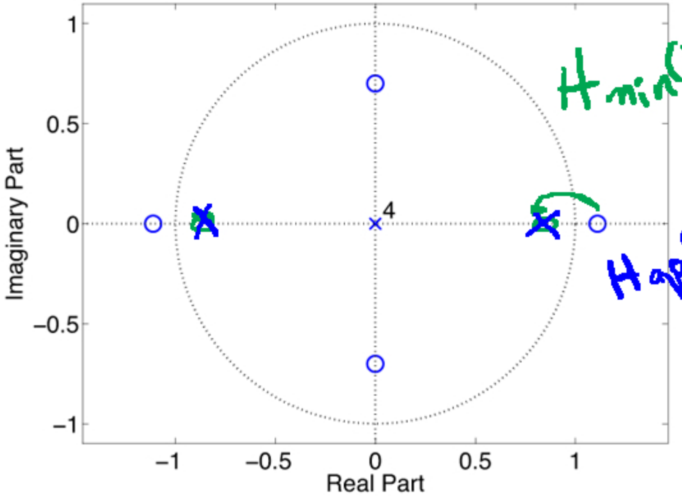


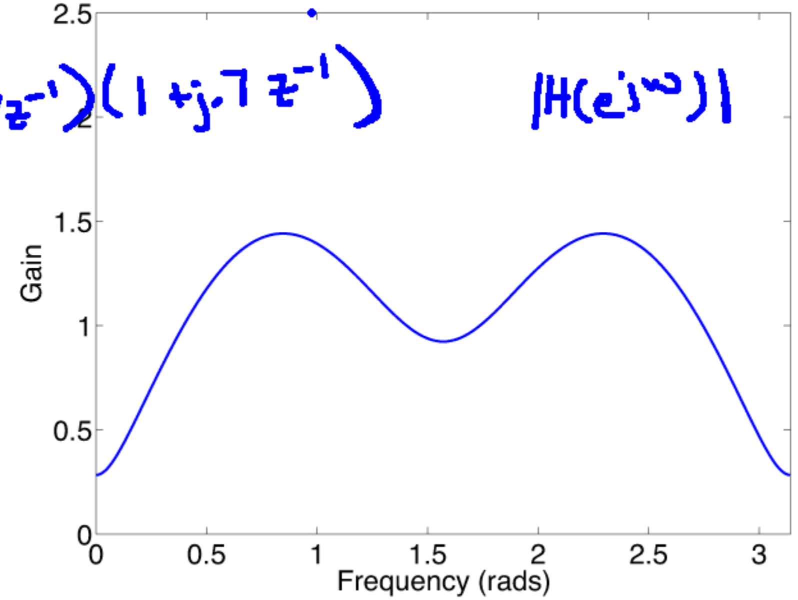
The minimum phase portion of any system has a stable, causal inverse system

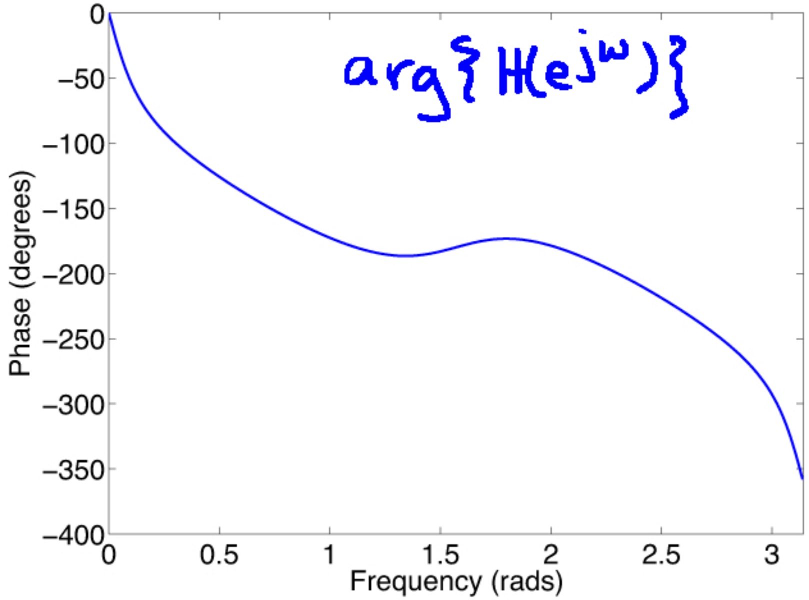


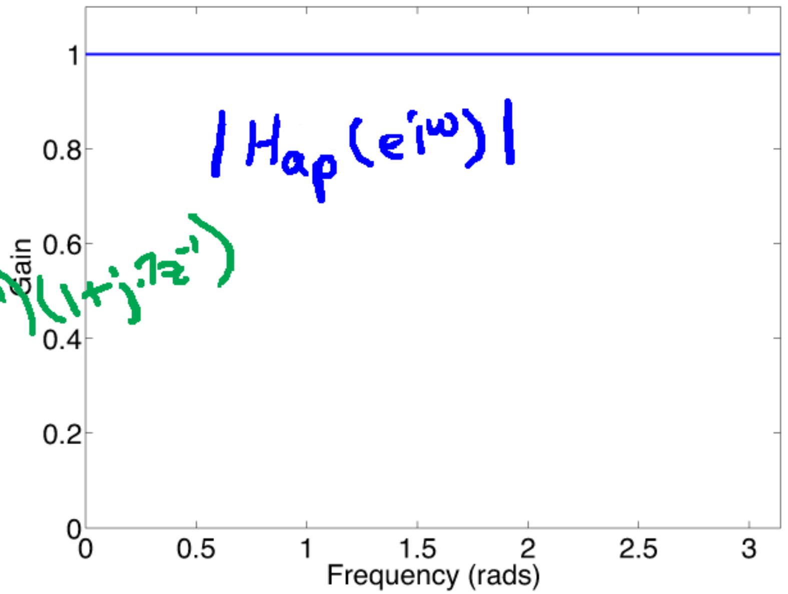
Example:

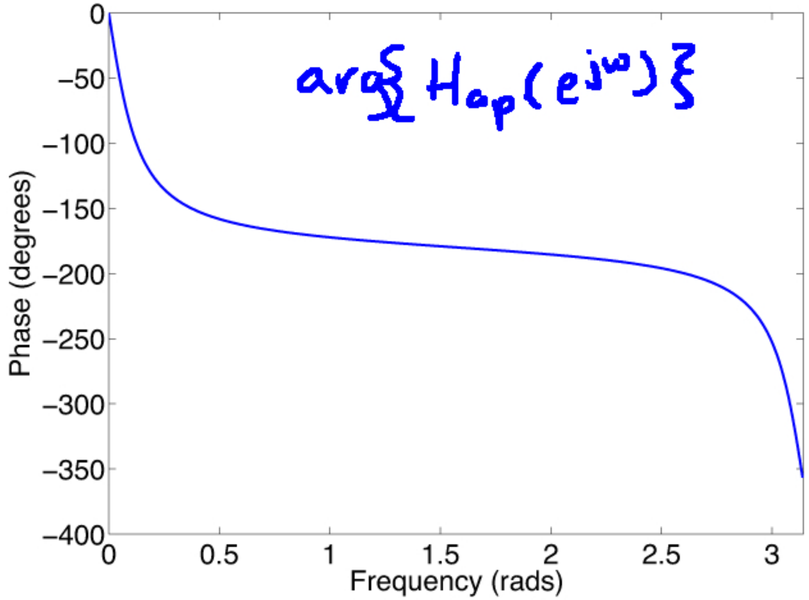
$$H(z) = \left(1 - \tfrac{1}{.9} z^{-1}\right)\left(1 + \tfrac{1}{.9} z^{-1}\right)\left(1 - j.7 z^{-1}\right)\left(1 + j.7 z^{-1}\right)$$











$$H_{ap}(z) = \frac{(z^{-1} - .9)(z^{-1} + .9)}{(1 - .9z^{-1})(1 + .9z^{-1})}$$

$$H_{min}(z) = .81(1 - .9z^{-1})(1 + .9z^{-1})(1 - j.7z^{-1})(1 + j.7z^{-1})$$