Inverse Systems for LTI Systems Described by Difference Equations

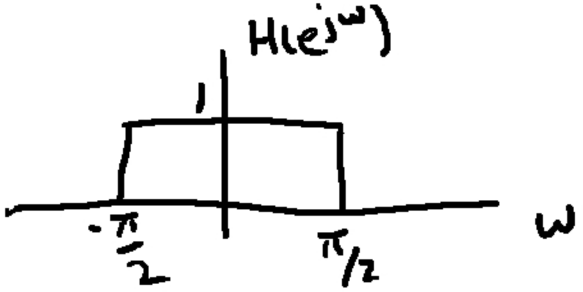
$$x[n] \to \boxed{H} \to y[n] \to \overset{?}{\boxed{H_I}} \to x[n]$$

$y[n] = h[n] * x[n]$, $x[n] = h_I[n] * y[n]$ $\Rightarrow h_I[n] * h[n] = \delta[n]$

$H(z)\,H_I(z) = 1 \Rightarrow H_I(z) = \dfrac{1}{H(z)}$ (ROCs must intersect)

For stability, ROC for $H_I(z)$ must include $|z| = 1$

$$\Rightarrow |H(z)|\Big|_{z=e^{j\omega}} > 0$$



$\Rightarrow$ no inverse system

LTI systems described by difference equations

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} = \frac{b_0}{a_0} \frac{\prod_{k=1}^{M}(1-c_k z^{-1})}{\prod_{k=1}^{N}(1-d_k z^{-1})}$$

zeros: c_k
poles: d_k

Cannot have $|c_k| = 1$ for stable $H_I(z)$

$$H_I(z) = \frac{a_0}{b_0} \frac{\prod_{k=1}^{N}(1-d_k z^{-1})}{\prod_{k=1}^{M}(1-c_k z^{-1})}$$

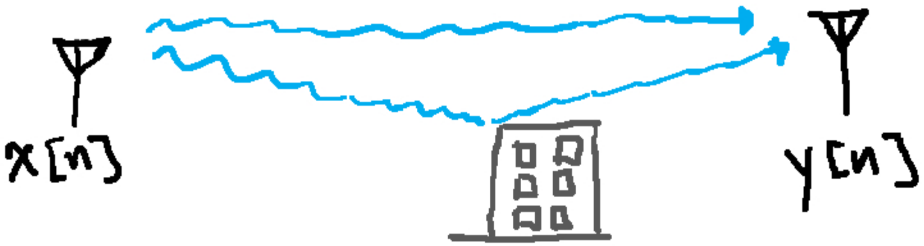
zeros: d_k
poles: c_k

ROC?

Stable/causal $H(z) \Longleftrightarrow |d_k| < 1$

Stable/causal $H_I(z) \Longleftrightarrow |c_k| < 1$

Example: Multipath Communication



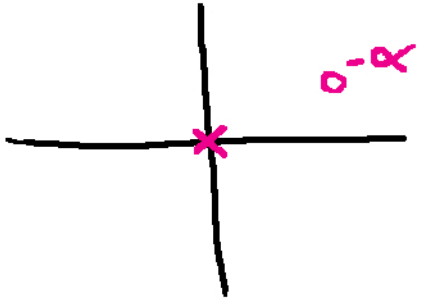
Diff. eqn model:

$$y[n] = x[n] + \alpha x[n-1]$$

Does a stable/causal inverse system exist?

$$H(z) = 1 + \alpha z^{-1}$$

pole: $z = 0$, zero: $z = -\alpha$

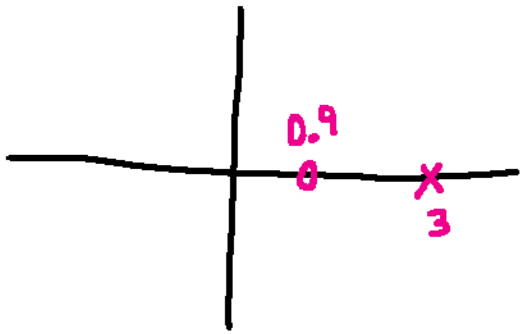


Require $|\alpha| < 1$!

$$H_I(z) = \frac{1}{1+\alpha z^{-1}} \implies y[n] + \alpha y[n-1] = x[n]$$

Example: $H(z) = \dfrac{z^{-1} - 1/3}{1 - 0.9z^{-1}}$ (stable/causal)

$$H_I(z) = \frac{1-0.9z^{-1}}{z^{-1}-1/3} = -3\,\frac{1-0.9z^{-1}}{1-3z^{-1}}$$ pole at $z = 3$



ROC $|z| < 3$ stable, not causal

$|z| > 3$ causal, not stable

A stable/causal inverse system does not exist!

If $H(z) = \dfrac{z^{-1} - 3}{1 - 0.9z^{-1}}$, $H_I(z) = -\dfrac{1}{3}\,\dfrac{1-0.9z^{-1}}{1-1/3\,z^{-1}}$ pole at $z = 1/3$

stable/causal inverse system exists!