Stability and Causality of LTI Systems Described by Difference Equations

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n] = x[n] * h[n]$$

Stable: $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$

Causal: $h[n] = 0 \quad , n < 0$

Difference Equation Description

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

ROC?

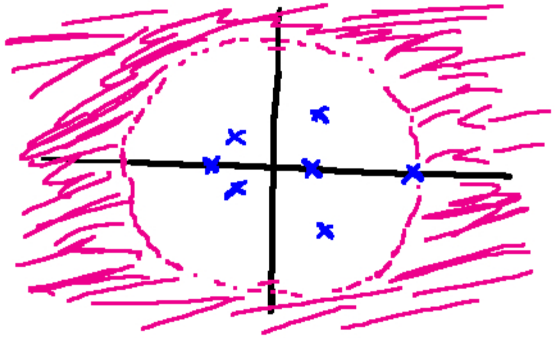
$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} \xleftrightarrow{z} h[n]$$

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} = \frac{b_0}{a_0} \frac{\prod_{k=1}^{M} (1 - c_k z^{-1})}{\prod_{k=1}^{N} (1 - d_k z^{-1})}$$

Causality: right-sided inverse transforms for all poles

ROC extends outward from pole with largest radius

$$|z| > \max_k |d_k|$$



Stability: absolute summability of $h[n]$

$$\infty > \sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=-\infty}^{\infty} |h[n]| |z|^{-n} \Big|_{|z|=1} = \sum_{n=-\infty}^{\infty} |h[n] z^{-n}| \Big|_{|z|=1}$$

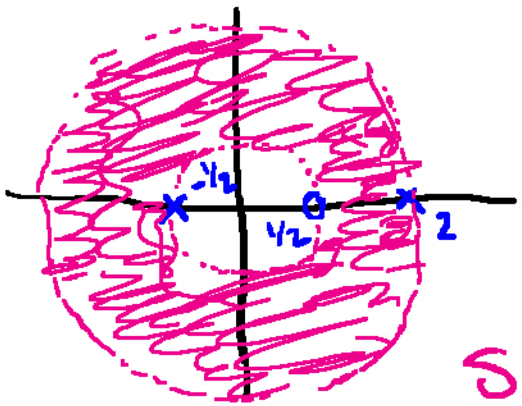
$$\geq \left| \sum_{n=-\infty}^{\infty} h[n] z^{-n} \right| \Big|_{|z|=1} = |H(z)|_{|z|=1}$$

$|H(z)|$ finite on $|z|=1$ $\Rightarrow$ ROC of $H(z)$ includes unit circle

Example: $y[n] - \frac{3}{2} y[n-1] - y[n-2] = 2x[n] - x[n-1]$

$$H(z) = \frac{2 - z^{-1}}{1 - \frac{3}{2} z^{-1} - z^{-2}} = \frac{2 - z^{-1}}{(1 + \frac{1}{2} z^{-1})(1 - 2z^{-1})}$$

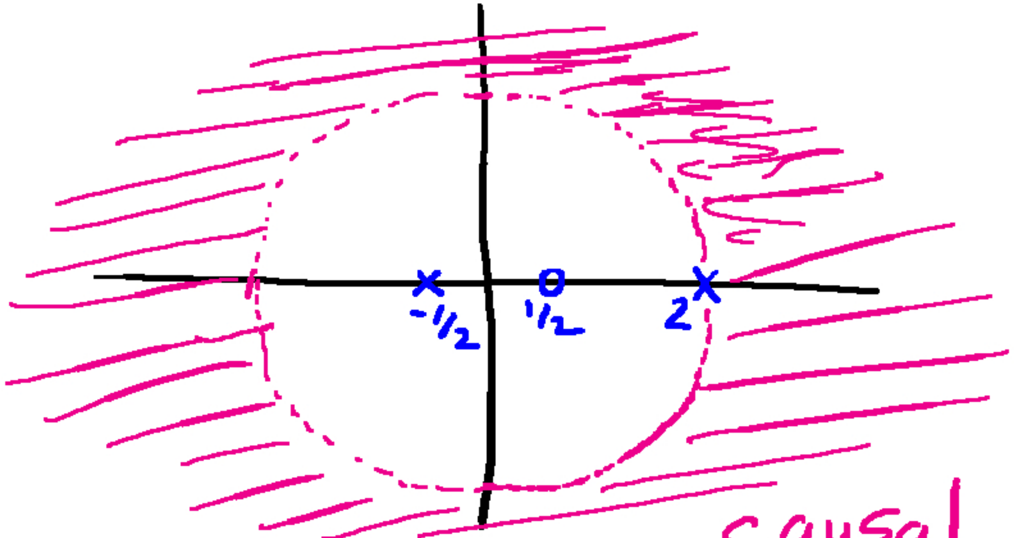
$$= \frac{2}{1 - 2z^{-1}} + \frac{1}{1 + \frac{1}{2} z^{-1}}$$



stable

$h[n] = \left(-\frac{1}{2}\right)^n u[n] - 2(2)^n u[-n-1]$

not causal!



causal

$h[n] = \left(-\frac{1}{2}\right)^n u[n] + 2(2)^n u[n]$

not stable!

Stable and causal $\Leftrightarrow$ all poles inside $|z|=1$

stability and causality are not always compatible

Examples

1) $y[n] + \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = x[n] - x[n-1]$

$$H(z) = \frac{1 - z^{-1}}{1 + \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}} = \frac{1 - z^{-1}}{(1 + \frac{1}{2}z^{-1})(1 + \frac{1}{4}z^{-1})}$$

poles: $z = -\frac{1}{2}, -\frac{1}{4}$

stable + causal

2) $$H(z) = \frac{z^2 + 2z + 1}{z - \frac{1}{2}}$$

poles: $z = \frac{1}{2}$, $z = \infty$

cannot be stable + causal

$$H(z) = z - 2 + \frac{7/2}{1 - \frac{1}{2}z^{-1}} \xleftrightarrow{z} h[n] = \underbrace{\delta[n+1]}_{n=-1} - 2\delta[n] + \frac{7}{2}\left(\frac{1}{2}\right)^n u[n]$$