The Inverse z-Transform: Partial Fraction Expansion

Partial Fraction Expansion –
inversion of rational functions of z^{-1}

Break $X(z)$ into elementary forms that can be inverted by inspection

$$X(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}} = \frac{8}{1 - z^{-1}} - \frac{9}{1 - \frac{1}{2}z^{-1}} + 2$$

$|z| > 1$

easy to invert

"undo" process of combining with a common denominator

$$X(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

write in powers of z^{-1}

1) If $M \geq N$ use long division to write

$$X(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \frac{\sum_{k=0}^{N-1} b'_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

2) Factor denominator as product of 1st order terms

$$\frac{\sum_{k=0}^{N-1} b'_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} = \frac{\sum_{k=0}^{N-1} b'_k z^{-k}}{a_0 \prod_{k=1}^{N} (1 - d_k z^{-1})}$$

3) Use partial fraction expansion in first-order terms

$$\frac{\sum_{k=0}^{N-1} b'_k z^{-k}}{a_0 \prod_{k=1}^{N} (1-d_k z^{-1})} = \sum_{k=1}^{N} \frac{A_k}{1-d_k z^{-1}} \quad \text{(assuming distinct } d_k\text{)}$$

if d_i repeated r times $\quad \sum_{\ell=1}^{r} \frac{A_i^{\ell}}{(1-d_i z^{-1})^{\ell}}$

4) Invert each term using ROC

$$\frac{A_k}{1-d_k z^{-1}} \overset{z}{\longleftrightarrow} A_k d_k^n u[n] \quad \text{for } |z|>|d_k|$$

$$\frac{A_k}{1-d_k z^{-1}} \overset{z}{\longleftrightarrow} -A_k d_k^n u[-n-1] \quad \text{for } |z|<|d_k|$$

$$\frac{A}{(1-d_k z^{-1})^{\ell}} \overset{z}{\longleftrightarrow} A \frac{(n+1)\cdots(n+\ell-1)}{(\ell-1)!} d_k^n u[n] \quad \text{for } |z|>|d_k|$$

$$\left(-A \frac{(n+1)\cdots(n+\ell-1)}{(\ell-1)!} d_k^n u[-n-1] \quad \text{for } |z|<|d_k| \right)$$

Example: $X(z) = \dfrac{1 - z^{-1} + z^{-2}}{(1-\frac{1}{2}z^{-1})(1-2z^{-1})(1-z^{-1})}$ ROC $1 < |z| < 2$

1) $M=2 < N=3 \Rightarrow$ no long division

2) Denominator is factored

3) Part Fraction Expansion

$$X(z) = \frac{A_1}{1-\frac{1}{2}z^{-1}} + \frac{A_2}{1-2z^{-1}} + \frac{A_3}{1-z^{-1}} \quad ; \quad A_1 = X(z)(1-\tfrac{1}{2}z^{-1})\Big|_{z^{-1}=2}$$

$$= \frac{1-z^{-1}+z^{-2}}{(1-2z^{-1})(1-z^{-1})}\Bigg|_{z^{-1}=2}$$

$$= \frac{1-2+4}{(1-4)(1-2)} = \frac{3}{3} = 1$$

$$A_2 = X(z)(1-2z^{-1})\Big|_{z^{-1}=1/2} = 2$$

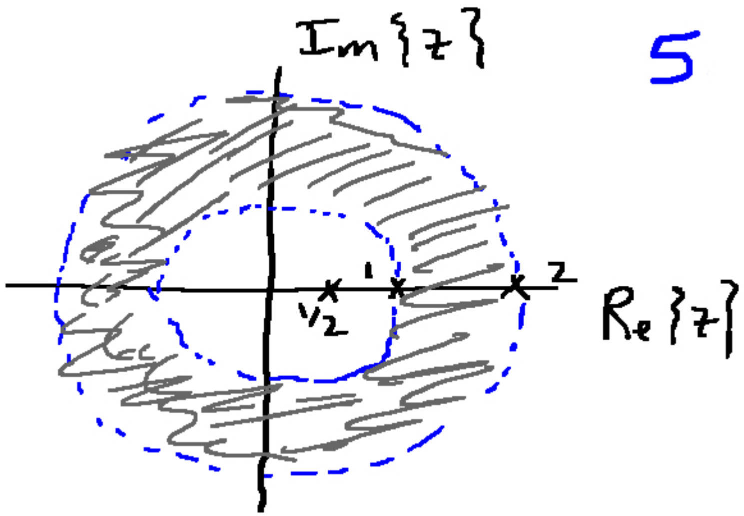
$$A_3 = X(z)(1-z^{-1})\Big|_{z^{-1}=1} = -2$$

$$X(z) = \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{2}{1-2z^{-1}} - \frac{2}{1-z^{-1}}$$

4) Invert each term using ROC info

5

ROC: $1 < |z| < 2$



$$\frac{1}{1-\frac{1}{2}z^{-1}} \xleftrightarrow{z} \left(\tfrac{1}{2}\right)^n u[n]$$

$$\frac{2}{1-2z^{-1}} \xleftrightarrow{z} -2\,(2)^n u[-n-1]$$

$$\frac{-2}{1-z^{-1}} \xleftrightarrow{z} -2\,u[n]$$

$$\underline{x[n] = \left(\tfrac{1}{2}\right)^n u[n] - 2\,(2)^n u[-n-1] - 2\,u[n]}$$

Example: $X(z) = \dfrac{z^3 - 10z^2 - 4z + 4}{2z^3 - 2z^2 - 4z}$ $|z| < 1$ 6

0) write in powers of z^{-1} $X(z) = \dfrac{1 - 10z^{-1} - 4z^{-2} + 4z^{-3}}{2 - 2z^{-1} - 4z^{-2}}$

1) $M=3 > N=2$ use long division

$$\begin{array}{r|l} & -z^{-1} + 3/2 \\ \hline -4z^{-2} - 2z^{-1} + 2 & 4z^{-3} - 4z^{-2} - 10z^{-1} + 1 \\ & 4z^{-3} + 2z^{-2} - 2z^{-1} \\ \hline & -6z^{-2} - 8z^{-1} + 1 \\ & -6z^{-2} - 3z^{-1} + 3 \\ \hline & -5z^{-1} - 2 \end{array}$$

$$X(z) = -z^{-1} + 3/2 + \frac{-5z^{-1} - 2}{2 - 2z^{-1} - 4z^{-2}}$$

2) Factor denominator

$$X(z) = -z^{-1} + 3/2 + \frac{-5z^{-1} - 2}{2(1+z^{-1})(1-2z^{-1})}$$

3) Partial fraction expansion

$$\frac{-5/2\, z^{-1} - 1}{(1+z^{-1})(1-2z^{-1})} = \frac{A_1}{1+z^{-1}} + \frac{A_2}{1-2z^{-1}}$$

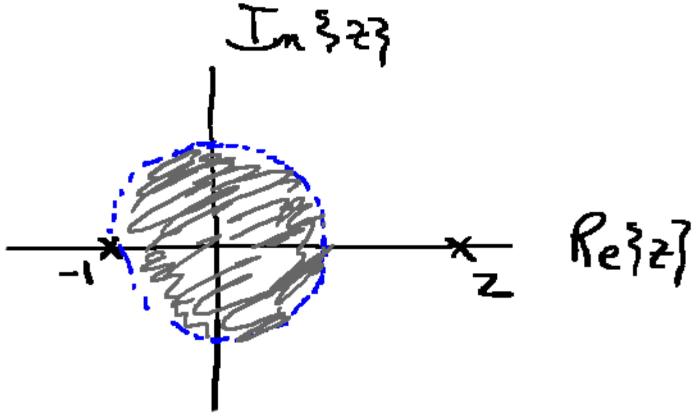
$$A_1 = \left.\frac{-\frac{5}{2}z^{-1} - 1}{1-2z^{-1}}\right|_{z^{-1}=-1} = 1/2$$

$$A_2 = \left.\frac{-\frac{5}{2}z^{-1} - 1}{1+z^{-1}}\right|_{z^{-1}=1/2} = -3/2$$

$$X(z) = -z^{-1} + 3/2 + \frac{1/2}{1+z^{-1}} - \frac{3/2}{1-2z^{-1}}, \quad |z| < 1$$

4) Invert terms using ROC

$$\underline{x[n] = -\delta[n-1] + 3/2\,\delta[n] - \frac{1}{2}(-1)^n u[-n-1] + \frac{3}{2}(2)^n u[-n-1]}$$



Remarks -

1) Poles of $X(z)$ with real coefficients occur in complex conjugate pairs: $d_1 = d_2^*$

PFE coefficients must be in complex conjugate pairs

$$\frac{A_1}{1-d_1 z^{-1}} + \frac{A_1^*}{1-d_2 z^{-1}} = \frac{A_1^R + jA_1^I}{1-d_1 z^{-1}} + \frac{A_1^R - jA_1^I}{1-d_2 z^{-1}}$$

2) Information other than ROC

- causal signal $\Rightarrow$ right-sided inverses
- stable signal $\Rightarrow$ ROC includes $|z| = 1$

$$\frac{z}{z-d} \longleftrightarrow d^n u[n]$$