Inverse z-Transform: Power Series Expansion

z-Transform is a power series

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

$$= \quad \cdots + x[-2]z^{2} + x[-1]z + x[0] + x[1]z^{-1} + x[2]z^{-2} + \cdots$$

1) Write function to be inverted as a power series

2) Identify $x[n]$ as coefficient of z^{-n}

Example:

$$X(z) = \underbrace{2}_{x[-5]} z^{5} + \underbrace{z^{3}}_{x[-3]} - \underbrace{z^{2}}_{x[-2]} + \underbrace{1}_{x[0]} + \underbrace{3}_{x[1]} z^{-1} - \underbrace{4}_{x[4]} z^{-4}$$

$$x[n] = \underline{2\delta[n+5] + \delta[n+3] - \delta[n+2] + \delta[n] + 3\delta[n-1] - 4\delta[n-4]}$$

Power series expansion can invert transcendental functions of z

Ex. $X(z) = \exp\{-2z^{-1}\}$

Recall $\exp\{x\} = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ so

$$X(z) = \sum_{n=0}^{\infty} \frac{(-2z^{-1})^n}{n!} = \sum_{n=0}^{\infty} \frac{(-2)^n}{n!} z^{-n}$$

$$x[n] = \frac{(-2)^n}{n!} u[n]$$

Can also invert rational $X(z)$ with long division

Ex. $X(z) = \dfrac{1 - z^{-1}}{1 - \frac{1}{2}z^{-1}}$

$|z| > \frac{1}{2}$

$$\begin{array}{r l}
 & 1 - \tfrac{1}{2}z^{-1} - \tfrac{1}{4}z^{-2} - \tfrac{1}{8}z^{-3} - \tfrac{1}{16}z^{-4} \cdots \\ \hline
1 - \tfrac{1}{2}z^{-1} \,\big) & 1 - z^{-1} \\
 & 1 - \tfrac{1}{2}z^{-1} \\ \hline
 & -\tfrac{1}{2}z^{-1} \\
 & -\tfrac{1}{2}z^{-1} + \tfrac{1}{4}z^{-2} \\ \hline
 & -\tfrac{1}{4}z^{-2} \\
 & -\tfrac{1}{4}z^{-2} + \tfrac{1}{8}z^{-3} \\ \hline
 & -\tfrac{1}{8}z^{-3} \\
 & -\tfrac{1}{8}z^{-3} + \tfrac{1}{16}z^{-4} \\ \hline
 & -\tfrac{1}{16}z^{-4}
\end{array}$$

$$x[n] = \begin{array}{ll}
0 & n < 0 \\
1 & n = 0 \\
-\tfrac{1}{2} & n = 1 \\
-\tfrac{1}{4} & n = 2 \\
-\tfrac{1}{8} & n = 3 \\
-\tfrac{1}{16} & n = 4 \\
\vdots &
\end{array}$$

$x[n] = 2\,\delta[n] - (\tfrac{1}{2})^n u[n]$