Poles and Zeros

Most useful and important z-transforms –

Rational functions $X(z) = \dfrac{P(z)}{Q(z)}$

with $P(z), Q(z)$: polynomials in z

zeros: values of z for which $X(z) = 0$

poles: values of z for which $X(z) = \infty$

roots of $P(z)$: zeros "o" roots of $Q(z)$: poles "x"

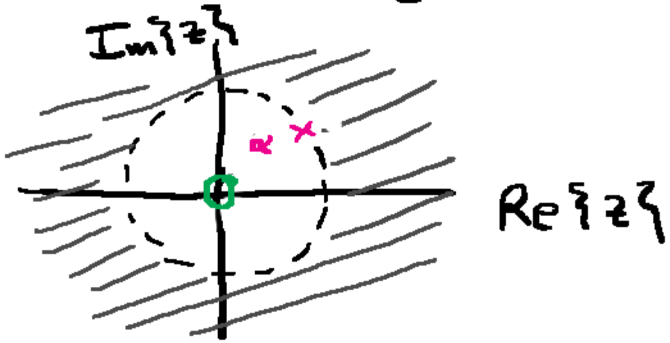
may also have poles/zeros at $z = \infty$ [order $Q(z) \neq$ order $P(z)$]

Examples:

1) $x[n] = \alpha^n u[n] \xleftrightarrow{z} X(z) = \dfrac{z}{z-\alpha}, \quad |z| > |\alpha|$

zero @ $z = 0$

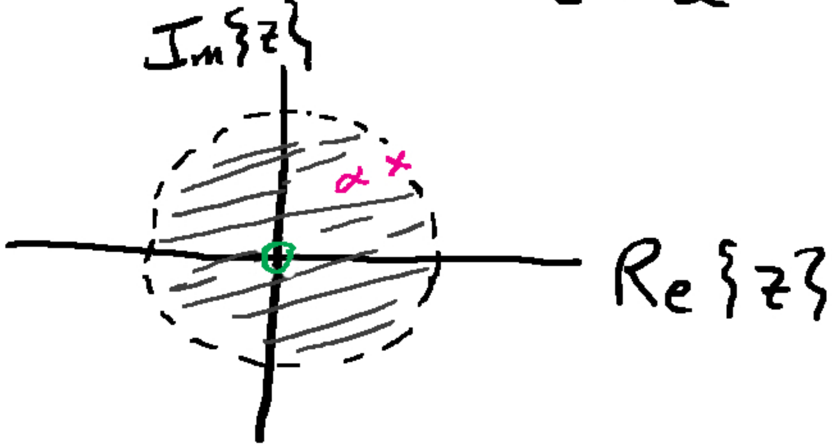
pole @ $z = \alpha$



2) $x[n] = -\alpha^n u[-n-1] \xleftrightarrow{z} X(z) = \dfrac{z}{z-\alpha}, \quad |z| < |\alpha|$

zero @ $z = 0$

pole @ $z = \alpha$



Examples-

3) $x[n] = \left(\frac{1}{4}\right)^n u[n] + \left(-\frac{1}{2}\right)^n u[n] \overset{z}{\longleftrightarrow}$

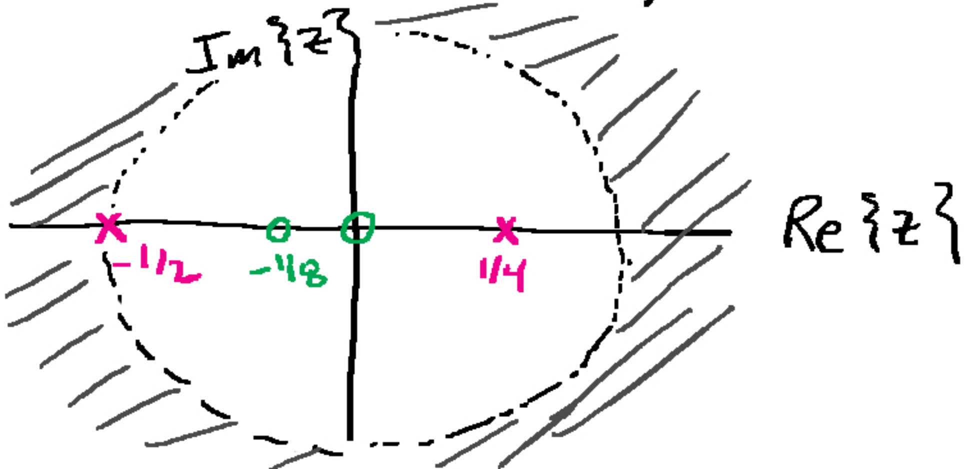
$$X(z) = \frac{z}{z - 1/4} + \frac{z}{z + 1/2}, \quad |z| > 1/2$$

$$X(z) = \frac{z^2 + 1/2 z + z^2 - 1/4 z}{(z - 1/4)(z + 1/2)}, \quad |z| > 1/2$$

$$= \frac{2z^2 + 1/4 z}{(z - 1/4)(z + 1/2)} = \frac{2z(z + 1/8)}{(z - 1/4)(z + 1/2)}, \quad |z| > 1/2$$

zeros @ $z = 0, -1/8$

poles @ $z = 1/4, -1/2$



Examples -

4) $x[n] = (1/4)^n u[n] - (-\frac{1}{2})^n u[-n-1] \overset{z}{\longleftrightarrow}$

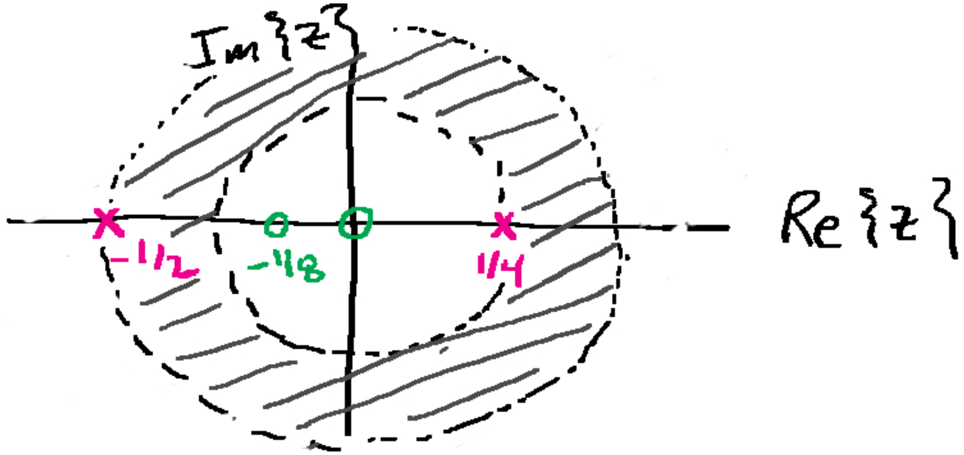
$$X(z) = \underbrace{\frac{z}{z-1/4}}_{|z|>1/4} + \underbrace{\frac{z}{z+1/2}}_{|z|<1/2}, \quad 1/4 < |z| < 1/2$$

from Ex. 3)

$$X(z) = \frac{2z(z+1/8)}{(z-1/4)(z+1/2)}, \quad 1/4 < |z| < 1/2$$

zeros @ $z = 0, -1/8$

poles @ $z = 1/4, -1/2$



Examples: 5) $x[n] = \begin{cases} \alpha^n & 0 \le n \le N-1 \\ 0 & \text{otherwise} \end{cases}$

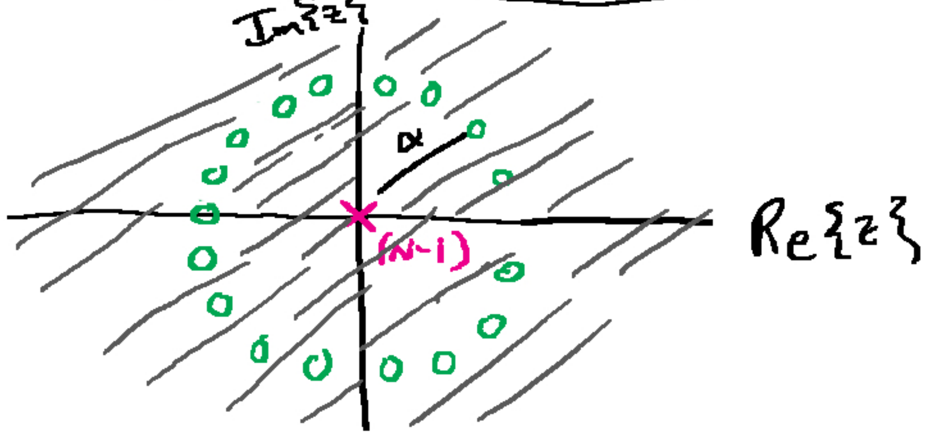
$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = \sum_{n=0}^{N-1} (\alpha z^{-1})^n = \frac{1 - (\alpha z^{-1})^N}{1 - \alpha z^{-1}} = \frac{z^N - \alpha^N}{z^{N-1}(z-\alpha)}$$

ROC: $\sum_{n=0}^{N-1} |\alpha z^{-1}|^n < \infty \Rightarrow |\alpha| < \infty$ and $z \neq 0$

(entire z-plane, except $z=0$)

Zeros: $z^N = \alpha^N \Rightarrow z_k = \alpha e^{j\frac{2\pi}{N}k}$, $k = \cancel{0}, 1, 2 \cdots N-1$

poles: $z=0$ $(N-1)$ and $z=\alpha$ — cancels out with zero $z_0 = \alpha$



Examples: poles & zeros at $z = \infty$

6) $X(z) = \dfrac{z+1}{(z+2)(z-1)}$

zero @ $z=-1$

poles @ $z=-2, 1$

BUT

$$\lim_{z\to\infty} X(z) \approx \lim_{z\to\infty} \frac{1}{z} = 0$$

$z=\infty$ is also a zero

7) $X(z) = \dfrac{(z+2)(z-1)}{z+1}$

zero @ $z=-2, 1$

pole @ $z=-1$

BUT

$$\lim_{z\to\infty} X(z) \approx \lim_{z\to\infty} z = \infty$$

$z=\infty$ is also a pole

If $X(z) = \dfrac{P(z)}{Q(z)}$ and order$\{P(z)\}=M$, order$\{Q(z)\}=N$ and

1) $N>M$, then

$N-M$ zeros @ $z=\infty$

2) $M>N$, then

$M-N$ poles @ $z=\infty$