Region of Convergence (ROC)

ROC: set of z for which the z transform of a signal $x[n]$ converges (exists)

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

Case 1: Delay $w[n] = \delta[n-n_0]$

$$W(z) = \sum_{n=-\infty}^{\infty} \delta[n-n_0] z^{-n} = z^{-n_0}$$

excludes $z=0$ for $n_0>0$
$z=\infty$ for $n_0<0$

$$\delta[n-n_0] \xleftrightarrow{z} z^{-n_0} \quad \forall \; \begin{array}{ll} z \neq 0 & \text{if } n_0 > 0 \\ z \neq \infty & \text{if } n_0 < 0 \end{array}$$

Case 2: $x[n] = \alpha^n u[n]$

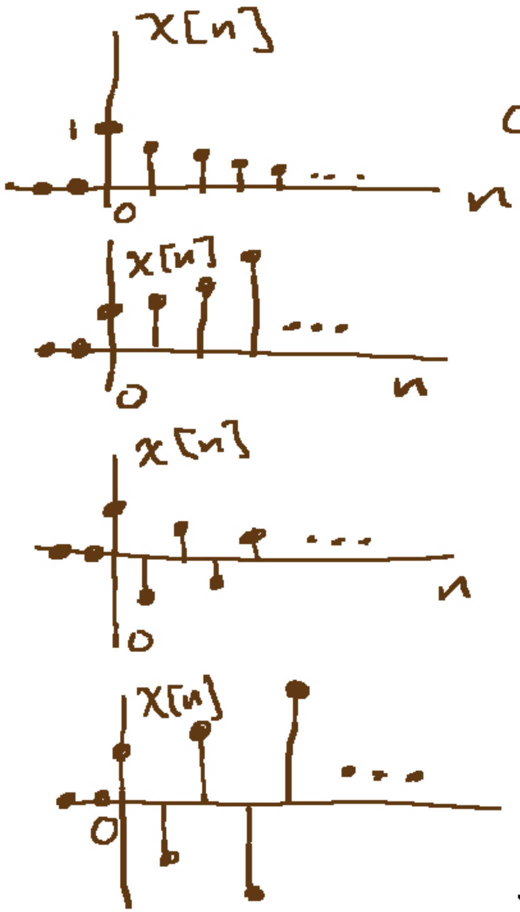
$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

$$= \sum_{n=0}^{\infty} \alpha^n z^{-n} = \sum_{n=0}^{\infty} (\alpha z^{-1})^n$$

$$= \frac{1}{1-\alpha z^{-1}} \qquad \text{provided } |\alpha z^{-1}| < 1$$

$$|z| > |\alpha|$$

if $|\alpha z^{-1}| \geq 1$, $X(z)$ does not converge



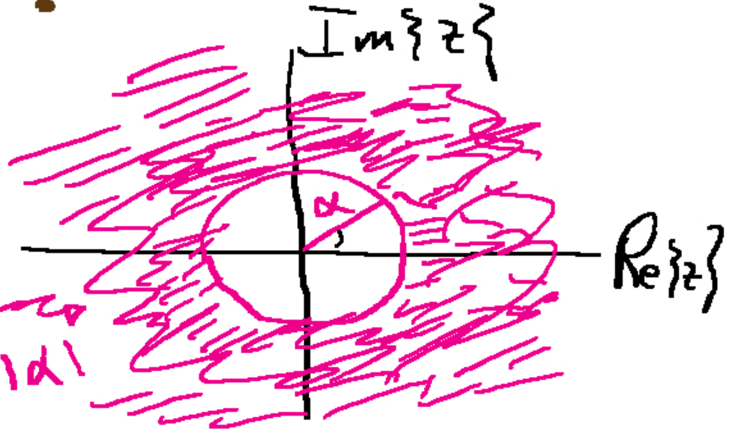
$0 < \alpha < 1$

$\alpha > 1$

$-1 < \alpha < 0$

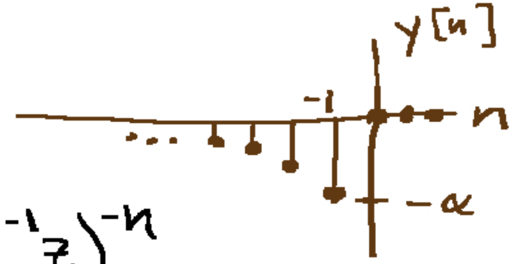
$\alpha < -1$

$$\alpha^n u[n] \xleftrightarrow{z} \frac{1}{1-\alpha z^{-1}} = \frac{z}{z-\alpha} \qquad |z| > |\alpha|$$



ROC: $|z| > |\alpha|$

Case 3: $y[n] = -\alpha^n u[-n-1]$



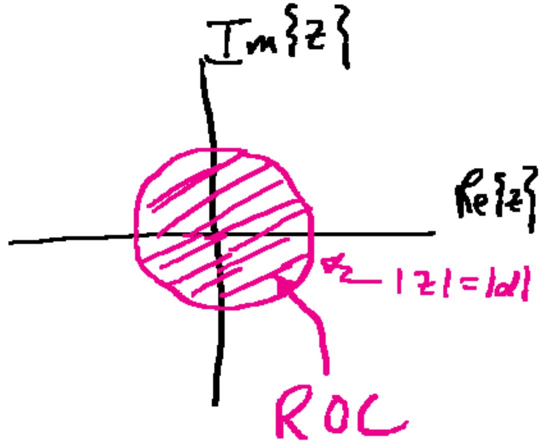
$0 < \alpha < 1$

$$Y(z) = \sum_{n=-\infty}^{\infty} y[n] z^{-n} = \sum_{n=-\infty}^{-1} -\alpha^n z^{-n} = -\sum_{n=-\infty}^{-1} (\alpha^{-1} z)^{-n}$$

$$= -\sum_{l=1}^{\infty} (\alpha^{-1} z)^{l} = -\sum_{l=0}^{\infty} (\alpha^{-1} z)^{l} + 1 = \frac{-1}{1-\alpha^{-1} z} + 1 \quad \text{for } |\alpha^{-1} z| < 1$$

$$= \frac{\alpha}{z-\alpha} + 1 = \frac{z}{z-\alpha} \quad \left(\text{or } \frac{1}{1-\alpha z^{-1}}\right) \qquad |z| < |\alpha|$$

$$-\alpha^n u[-n-1] \overset{z}{\longleftrightarrow} \boxed{\frac{1}{1-\alpha z^{-1}} = \frac{z}{z-\alpha}} \quad |z| < |\alpha|$$



X(z) (case 2) is identical to Y(z)!

<u>Differ in the ROC</u> - z transforms nonunique w/o ROC

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

power series in z

Convergence $\Rightarrow$ absolutely summable

Require $\displaystyle\sum_{n=-\infty}^{\infty} \left| x[n] z^{-n} \right| = \sum_{n=-\infty}^{\infty} |x[n]| \, |z|^{-n} < \infty$

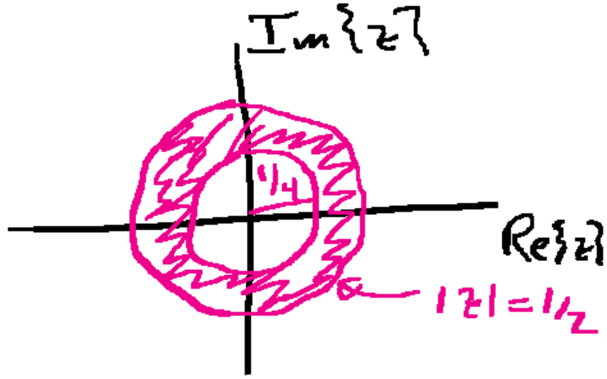
ROC depends on $|z|$ – circles/rings in z-plane

Example: $g[n] = \left(\frac{1}{4}\right)^n u[n] - \left(\frac{1}{2}\right)^n u[-n-1]$

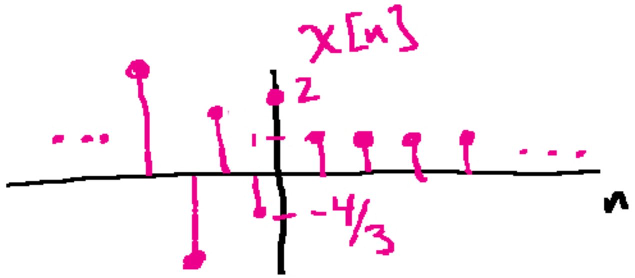
$$\left.\begin{aligned} \left(\tfrac{1}{4}\right)^n u[n] &\overset{z}{\longleftrightarrow} \frac{z}{z-\frac{1}{4}}, \quad |z| > \tfrac{1}{4} \\ -\left(\tfrac{1}{2}\right)^n u[-n-1] &\overset{z}{\longleftrightarrow} \frac{z}{z-\frac{1}{2}}, \quad |z| < \tfrac{1}{2} \end{aligned}\right\} \rightarrow$$

$$G(z) = \frac{z}{z-\frac{1}{4}} + \frac{z}{z-\frac{1}{2}}, \quad \tfrac{1}{4} < |z| < \tfrac{1}{2}$$

$$= \frac{2z^2 - \frac{3}{4}z}{(z-\frac{1}{4})(z-\frac{1}{2})},$$



Example: $x[n] = u[n] + \left(-\tfrac{3}{4}\right)^n u[-n]$



To use transform pairs, rewrite

$$x[n] = u[n] + \delta[n] - -\left(-\tfrac{3}{4}\right)^n u[-n-1]$$

$$u[n] \overset{z}{\longleftrightarrow} \frac{z}{z-1}, \quad |z| > 1$$

$$\delta[n] \overset{z}{\longleftrightarrow} 1$$

$$-\left(-\tfrac{3}{4}\right)^n u[-n-1] \overset{z}{\longleftrightarrow} \frac{z}{z+\tfrac{3}{4}}, \quad |z| < \tfrac{3}{4}$$

incompatible

$X(z)$ does not exist!