2-D Discrete Space Fourier Methods

A) Discrete Space FT

$$F(u,v) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f[m,n]\, e^{-jum} e^{-jvn}$$

u: vert. freq.
v: horiz. freq.
(rads)

- properties analogous to DTFT

$$g[m,n] = h[m,n] * f[m,n] \xleftrightarrow{\text{DSFT}} G(u,v) = H(u,v)\,F(u,v)$$

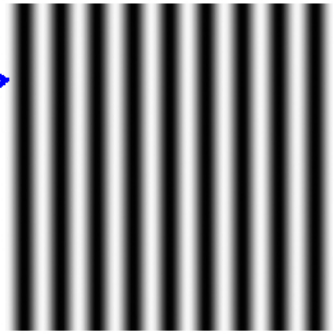
convolution in space

multiplication in frequency

2-D Sinusoids

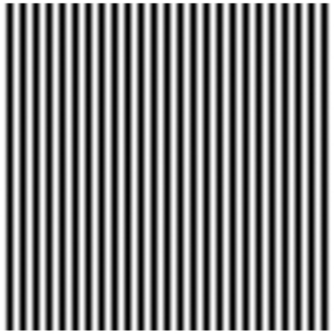
$\rightarrow n$

$m \downarrow$



$\cos\left(\frac{18\pi}{256} n\right)$

$u=0, \; v=\frac{18\pi}{256}$

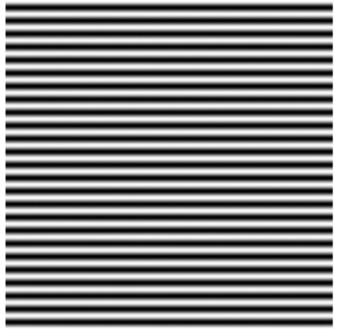


$\cos\left(\frac{50\pi}{256} n\right)$

$u=0, \; v=\frac{50\pi}{256}$

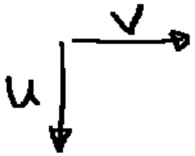
$\cos\left(\frac{18\pi}{256} m\right)$

$u=\frac{18\pi}{256}, \; v=0$



$\cos\left(\frac{50\pi}{256} m\right)$

$u=\frac{50\pi}{256}, \; v=0$



$\cos\left(\frac{50\pi}{256} n\right) \cos\left(\frac{18\pi}{256} m\right)$

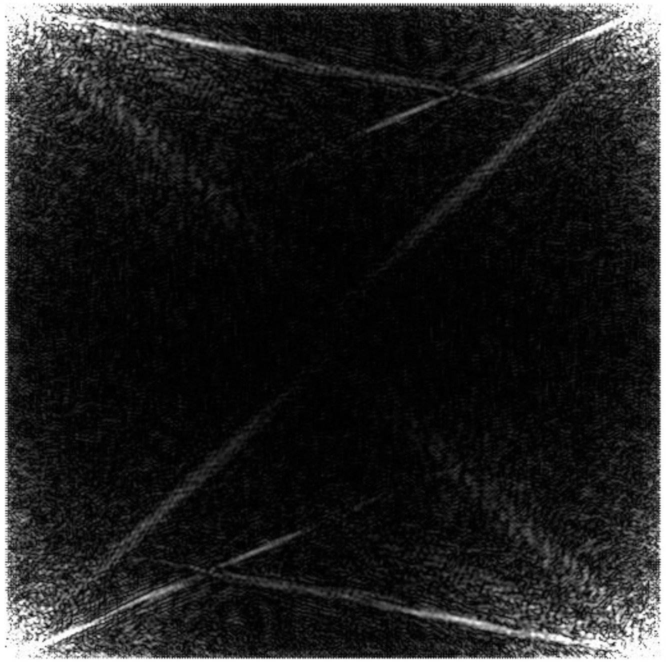
$u=\frac{18\pi}{256}, \quad v=\frac{50\pi}{256}$

$e^{-jum} \; e^{-jvn}$

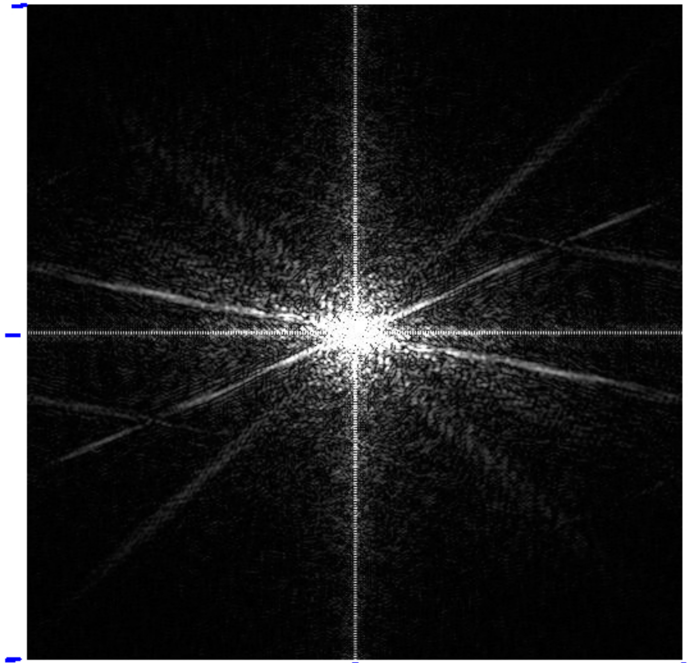
Example: Cameraman Image

$f(m,n)$

$|F(u,v)|$



$|F(u,v)|$



2-D Discrete Fourier Transform

$$F[k,l] = F(u,v)\Big|_{u=\frac{2\pi}{N}k,\ v=\frac{2\pi}{N}l} \qquad \begin{array}{l} k = 0,1,\dots N-1 \\ l = 0,1,\dots N-1 \end{array}$$

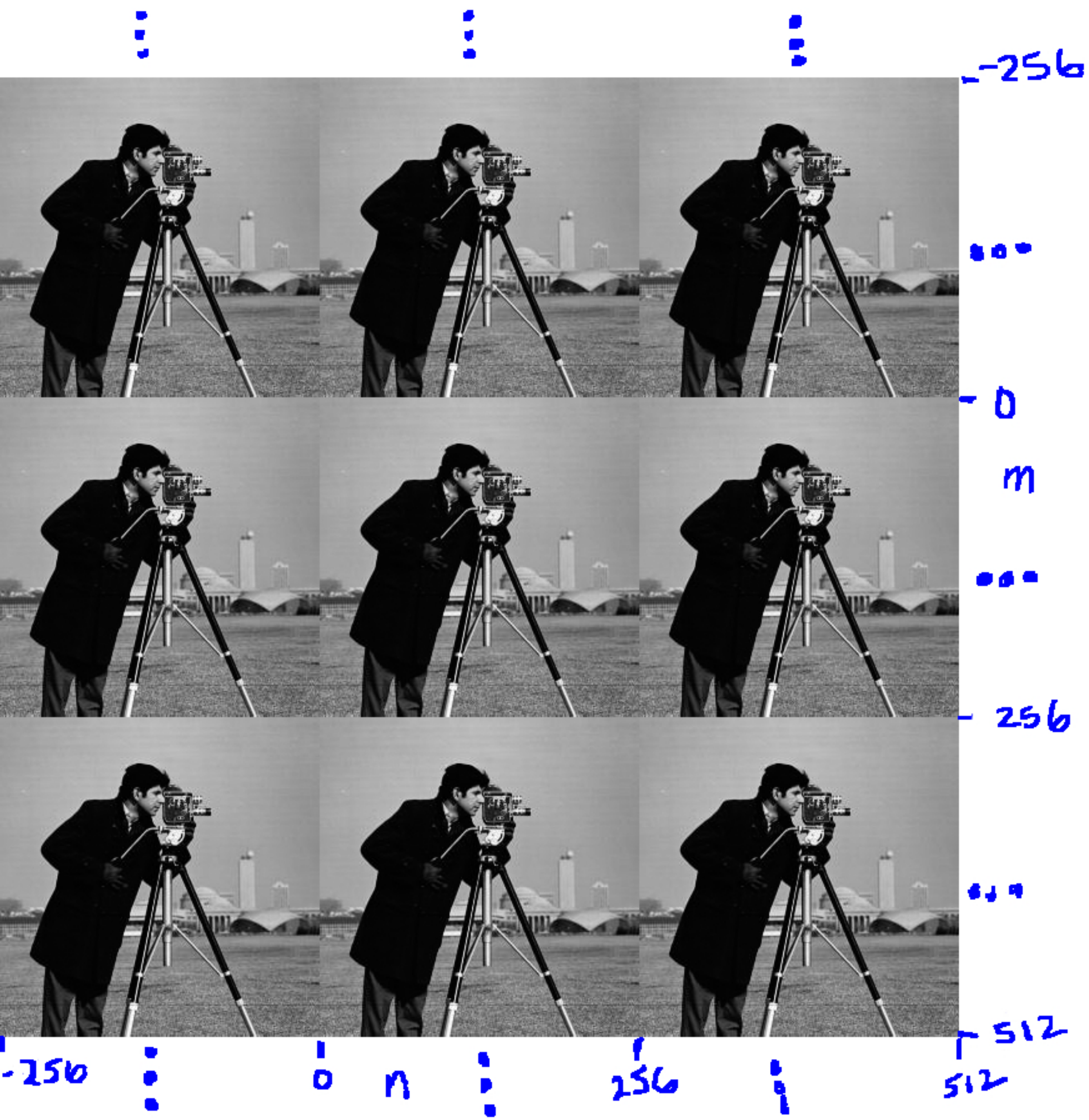
(assumes $N \times N$ image)

$$F[k,l] = \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} f[m,n]\, e^{-j\underline{\frac{2\pi}{N}}km}\, e^{-j\underline{\frac{2\pi}{N}}ln}$$ 2D DFT

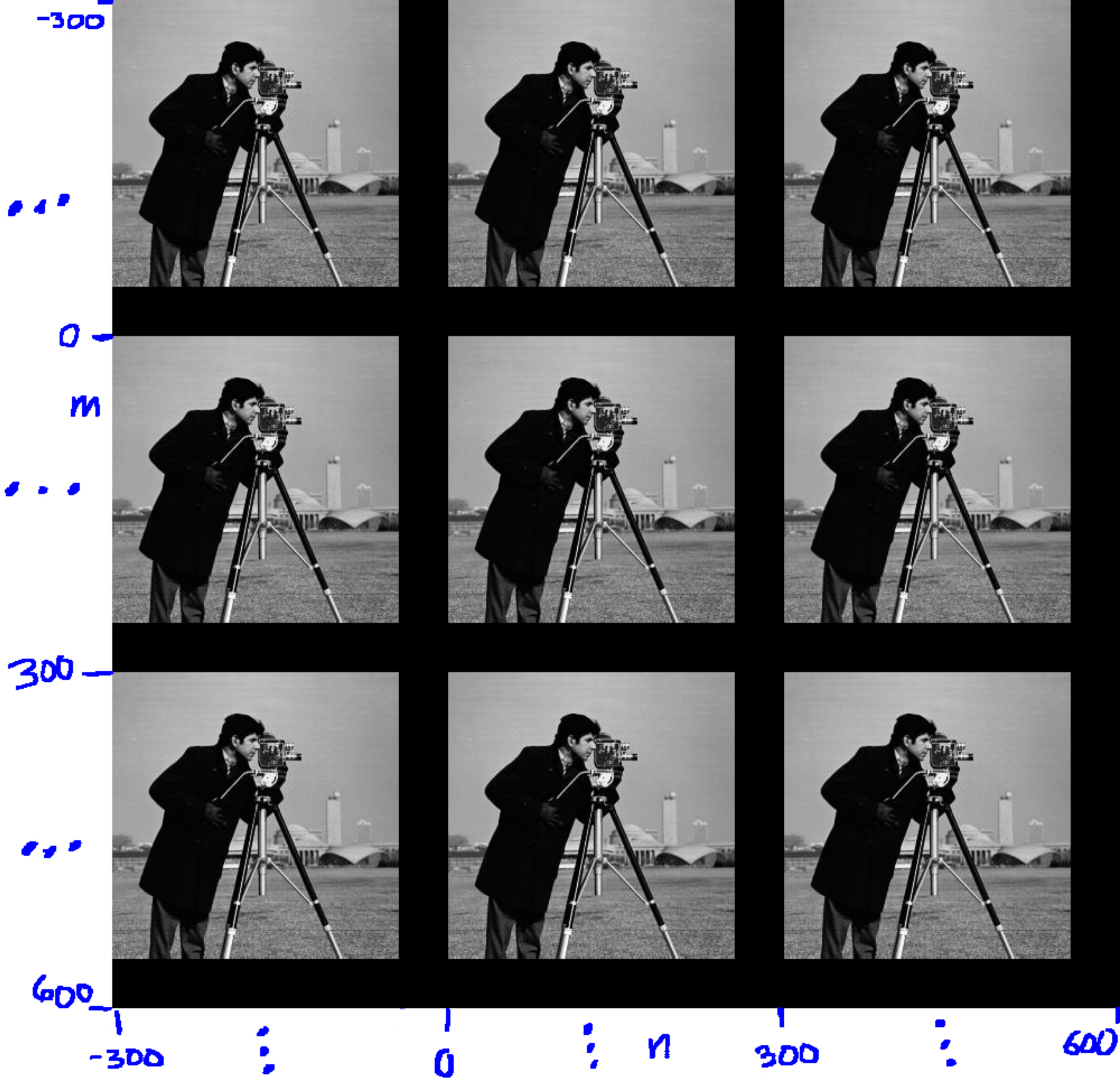
$$f[m,n] = \frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} F[k,l]\, \underline{e^{j\frac{2\pi}{N}km}}\, \underline{e^{j\frac{2\pi}{N}ln}}$$ 2D Inv. DFT

Sampling in frequency at $\frac{2\pi}{N}$ introduces N periodicity in $[m,n]$

Image 256×256



N=256 freq. samples



N=300 freq. samples

DFT Convolution - Multiplication

$$H[k,l]F[k,l] \xleftrightarrow{\text{2D DFT in } N} h[m,n] \circledast_N f[m,n]$$

multiplication

2D circular convolution

(N periodic extension due to freq. sampling)

Obtain linear convolution with zero padding!

if $\underline{h}$ is $M \times M$, $\underline{f}$ is $P \times P$, choose $N \geq M+P-1$

then $h[m,n] * f[m,n] = h[m,n] \circledast_N f[m,n]$

zero pad to $N \geq M+P-1$

2D DFTs

Multiply DFT coeffs

2D Inv DFT

Computing the 2D DFT

$$F[k,l] = \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} f[m,n]\, e^{-j\frac{2\pi}{N}km}\, e^{-j\frac{2\pi}{N}ln}$$

$$= \sum_{m=0}^{N-1} \underbrace{\left(\sum_{n=0}^{N-1} f[m,n]\, e^{-j\frac{2\pi}{N}ln} \right)}_{\text{1-D DFT over } n} e^{-j\frac{2\pi}{N}km}$$

use 1-D FFT for each m

← N values of m

$O(N^2 \log_2 N)$

$$= \sum_{m=0}^{N-1} \tilde{f}[m,l]\, e^{-j\frac{2\pi}{N}km}$$

1-D FFT for each value l

N values of l

$> O(N^2 \log_2 N)$

Overall complexity $\sim O(N^2 \log_2 N)$

← total # pixels