Using the DFT to Approximate the FT

- Numerical analysis of the frequency content of continuous-time signals

Three Approximation Steps

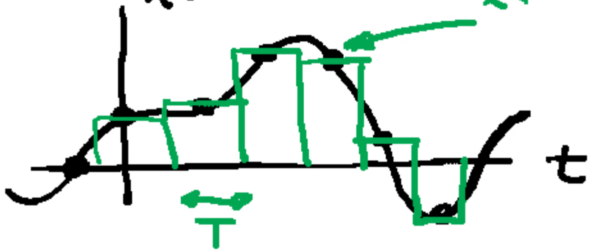
1) Sample in time
2) Truncate in time
3) Sample in frequency

$$\text{FT} \downarrow \text{DFT}$$

FT : $$X(\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}\,dt$$

1) Sample in time : Riemann sum approximation

$$\hat{X}(\Omega) = \sum_{n=-\infty}^{\infty} x(nT)e^{-j\Omega nT}\;T$$



$$= T\sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega Tn}$$

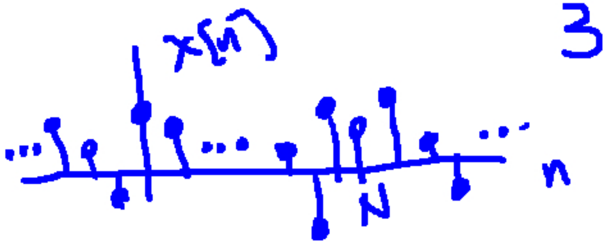
$$= T\;X(e^{j\omega})\Big|_{\omega=\Omega T}$$ DTFT

Accuracy: a) Sampling Theorem

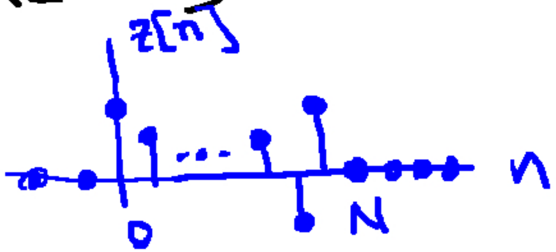
b) Approx improves as T ↓

2) Truncate to N samples

$$\hat{X}_N(\Omega) = T \sum_{n=0}^{N-1} x[n] e^{-j\Omega T n}$$



Let $z[n] = \begin{cases} x[n], & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases} \xleftrightarrow{\text{DTFT}} Z(e^{j\omega})$



$$\hat{X}_N(\Omega) = T\, Z(e^{j\omega})\Big|_{\omega = \Omega T}$$

But $z[n] = x[n]w[n]$ where $w[n] = \begin{cases} 1, & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$

$\Big\updownarrow$ DTFT

$$W(e^{j\omega}) = e^{-j(N-1)\omega/2} \frac{\sin\left(\frac{N\omega}{2}\right)}{\sin\left(\frac{\omega}{2}\right)}$$

$$Z(e^{j\omega}) = \frac{1}{2\pi} X(e^{j\omega}) * W(e^{j\omega})$$

distorted version of $X(e^{j\omega})$

3) Sample frequency at $\omega_k = \frac{2\pi}{M} k \quad (M \geq N)$

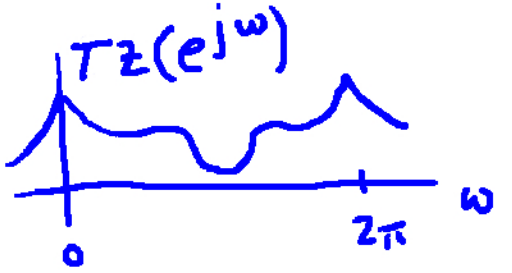
$$\Omega_k = \frac{2\pi}{MT} k$$

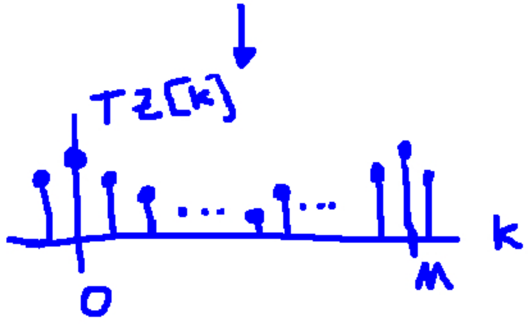
$$\hat{X}_N(\Omega_k) = T \sum_{n=0}^{N-1} x[n] e^{-j\left(\frac{2\pi}{MT}\right) k T n}$$

$$= T \sum_{n=0}^{N-1} x[n] e^{-j \frac{2\pi}{M} k n}$$

$$= T \; Z(e^{j\omega}) \Big|_{\omega_k = \frac{2\pi}{M} k}$$

$$= T \, Z[k] \quad \text{where} \quad z[n] \xleftrightarrow{\text{DFT};\, M} Z[k]$$

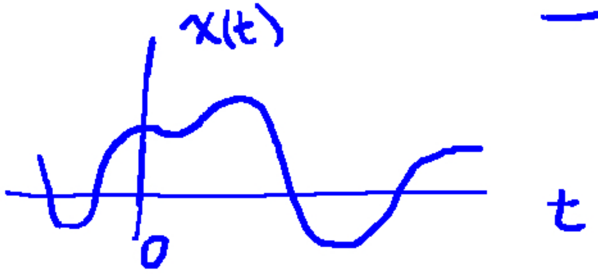




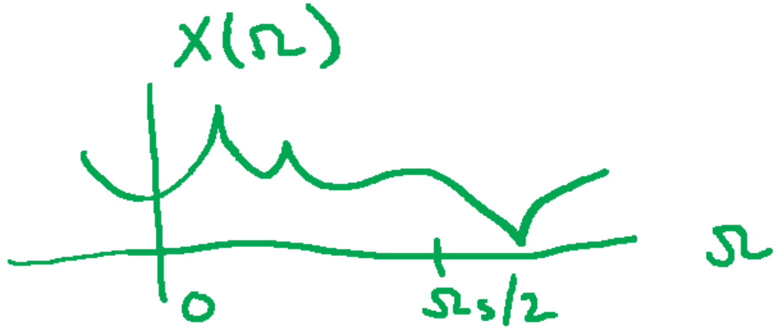
Increase M (zero padding) $\Rightarrow$ sample $Z(e^{j\omega})$ more densely

Does not eliminate distortion due to $W(e^{j\omega})$

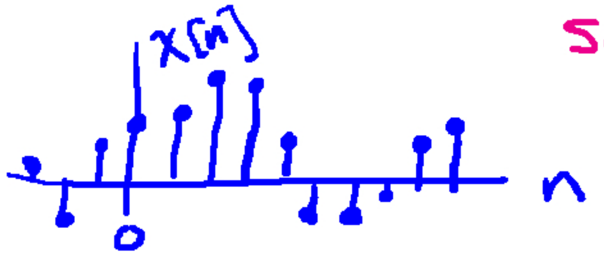
Summary



$\xleftrightarrow{\text{FT}}$



sampling ⇒ limits bandwidth



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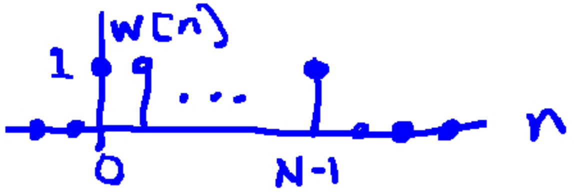
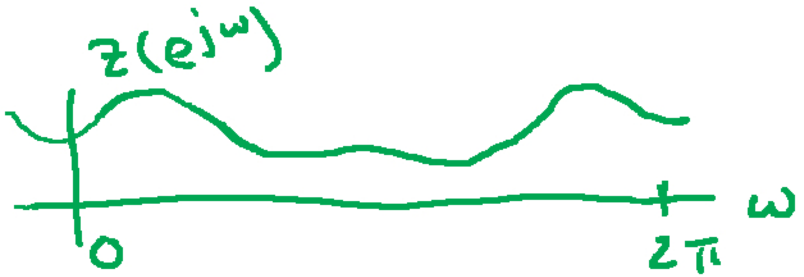


truncation/windowing

⇒ smoothing/distortion



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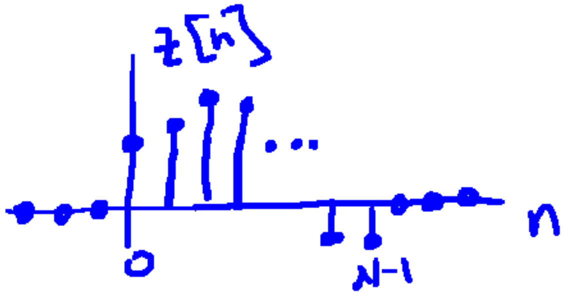

$\xleftrightarrow{\text{DFT};\ M}$