DFT Convolution Property

Convolution $\xleftrightarrow[?]{DFT}$ Multiplication

Consider $\tilde{Y}[k] = X[k]H[k]$ where $x[n] \xleftrightarrow{DFT;N} X[k]$, $h[n] \xleftrightarrow{DFT;N} H[k]$

$$\tilde{y}[n] = \frac{1}{N}\sum_{k=0}^{N-1} H[k]X[k]e^{j\frac{2\pi}{N}kn}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1}\left(\sum_{m=0}^{N-1} h[m]e^{-j\frac{2\pi}{N}km}\right)X[k]e^{j\frac{2\pi}{N}kn}$$

$$= \sum_{m=0}^{N-1} h[m]\underbrace{\left[\underbrace{\left(\frac{1}{N}\sum_{k=0}^{N-1}X[k]e^{j\frac{2\pi}{N}kn}\right)}_{x[((n))_N]}e^{-j\frac{2\pi}{N}km}\right]}_{x[((n-m))_N]}$$

$$= \sum_{m=0}^{N-1} h[m]\,x[((n-m))_N]$$

$$\tilde{Y}[k] = H[k]X[k] \xleftrightarrow{DFT;N} \tilde{y}[n] = \sum_{m=0}^{N-1} h[m]\, x[((n-m))_N]$$

$$= h[n] \circledN x[n]$$

Ⓝ "circular convolution": convolve $h[n]$ with an N-periodic extension of $x[n]$

$$h[n] \circledN x[n] \xleftrightarrow{DFT;N} H[k]X[k]$$

In general —

$$h[n] \circledN x[n] \neq h[n] * x[n]$$

Since $x[((n))_N] = \sum_{\ell=-\infty}^{\infty} x[n-\ell N]$

$$\tilde{y}[n] = h[n] \circledast x[n] = h[n] * x[((n))_N]$$

$$= h[n] * \sum_{\ell=-\infty}^{\infty} x[n-\ell N]$$

$$= \sum_{\ell=-\infty}^{\infty} h[n] * x[n-\ell N]$$

let $y[n] = h[n] * x[n]$

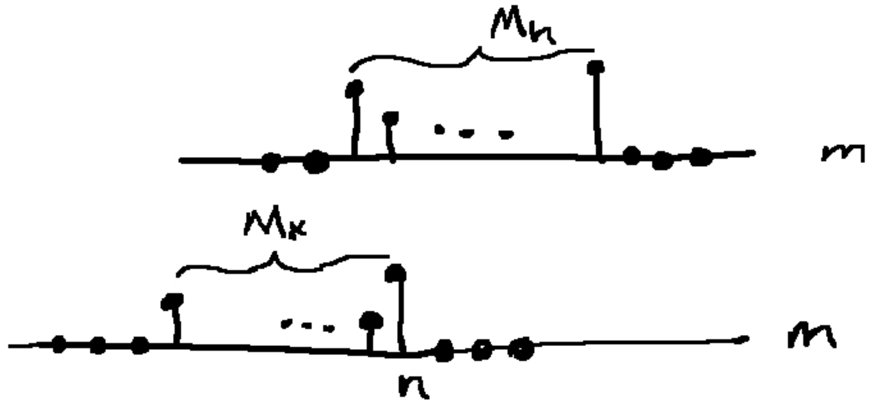
$$= \sum_{\ell=-\infty}^{\infty} y[n-\ell N]$$

sum of shifted replicates

Shifted replicates overlap if duration $y[n] > N$

Can obtain $y[n]$ from $\tilde{y}[n]$ iff duration $y[n] \le N$

If $x[n]$ is duration M_x and $h[n]$ is duration M_h,
then $y[n] = h[n] * x[n]$ is duration $M_y = M_x + M_h - 1$



Choose $N \geq M_x + M_h - 1$ to obtain $y[n] = h[n] * x[n]$

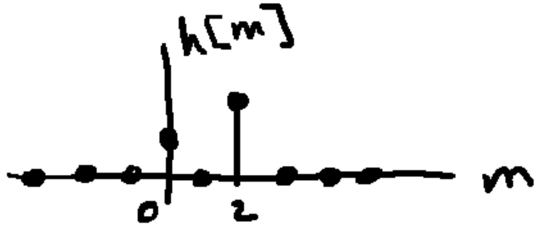
from $\tilde{y}[n] = h[n] \circledast x[n] \xleftrightarrow{DFT;N} H[k]X[k]$

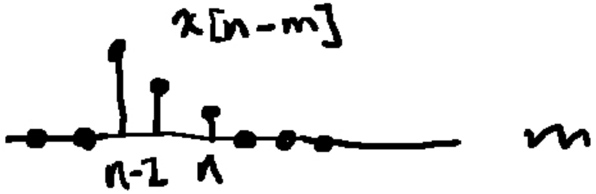
Example:

$$\tilde{y}[n] = h[n] \; Ⓝ \; x[n], \quad N = 3$$

$$= h[n] * x[((n))_3]$$

$$y[n] = h[n] * x[n] = \sum_m h[m] x[n-m]$$





$y[n] = 0, \quad n < 0$

$y[0] = 1$

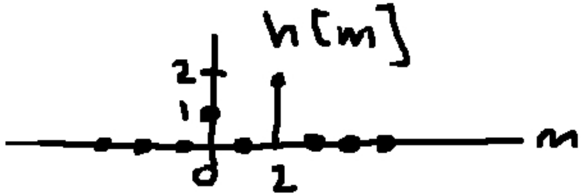
$y[1] = 2$

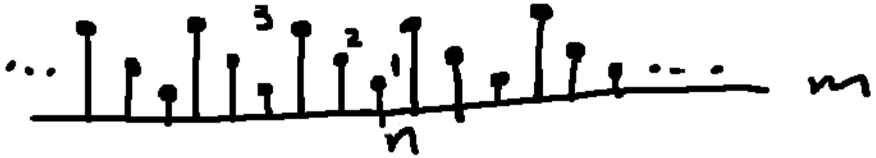
$y[2] = 5$

$y[3] = 4$

$y[4] = 6$

$y[n] = 0 \quad n \geq 5$





$\tilde{y}[n] \neq 0 \quad n < 0$

$\tilde{y}[0] = 5$

$\tilde{y}[1] = 8$

$\tilde{y}[2] = 5$

$\tilde{y}[3] = 5$

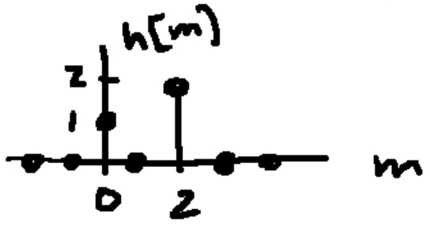
$\tilde{y}[4] = 8$

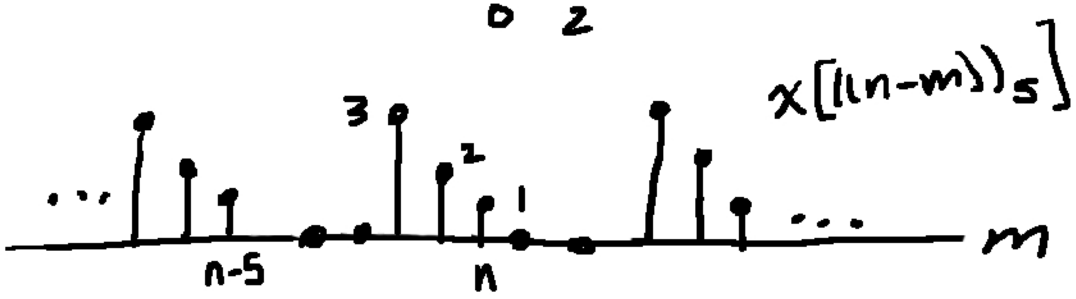
$\tilde{y}[5] = 5$

$\vdots$

Now consider $h[n] \circledast x[n]$ for $N = 3+3-1 = 5$

$$\tilde{y}[n] = h[n] * x[((n))_5]$$





$\tilde{y}[n] \neq 0 \quad n < 0$

$\tilde{y}[0] = 1$

$\tilde{y}[1] = 2$

$\tilde{y}[2] = 5$

$\tilde{y}[3] = 4$

$\tilde{y}[4] = 6$

$\vdots$

Here $y[n]$ has duration 5 $(y[0], y[1], \dots y[4] \neq 0)$ and for $N=5$, $y[n] = \tilde{y}[n]$, $n = 0, 1, 2, 3, 4$

Can recover $y[n]$ from $\tilde{y}[n]$.