The Discrete Fourier Transform (DFT)

Computer-based frequency domain analysis

- Spectral Analysis (e.g. finding periodicities)
- Denoising
- Compression (e.g. JPEG)
- Filtering
- "Fast" Convolution

Computer Analysis: discrete time, finite duration (N)

DTFT: $$X(e^{j\omega}) = \sum_{n=0}^{N-1} x[n]\, e^{-j\omega n}$$

– sample ω at $\omega_k = \frac{2\pi}{N} k$, $k = 0, 1, 2, \ldots, N-1$

Define $$X[k] = X(e^{j\frac{2\pi}{N}k})$$

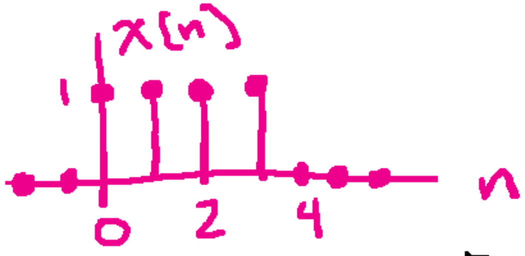
$$= \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}kn}$$

DFT of $x[n]$

Inverse DFT (IDFT)

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k]\, e^{j\left(\frac{2\pi}{N}k\right)n}$$

Example:



$N=4$

DTFT

$$X(e^{j\omega}) = \sum_{n=0}^{3} 1\cdot e^{-j\omega n}$$

$$= \frac{1-e^{-j4\omega}}{1-e^{-j\omega}}$$

$$= e^{-j3\omega/2}\,\frac{\sin(2\omega)}{\sin(\omega/2)}$$

$$X(e^{j\frac{2\pi}{4}k}) = e^{-j3\pi/4\,k}\,\frac{\sin(\pi k)}{\sin(\pi/4\,k)}$$

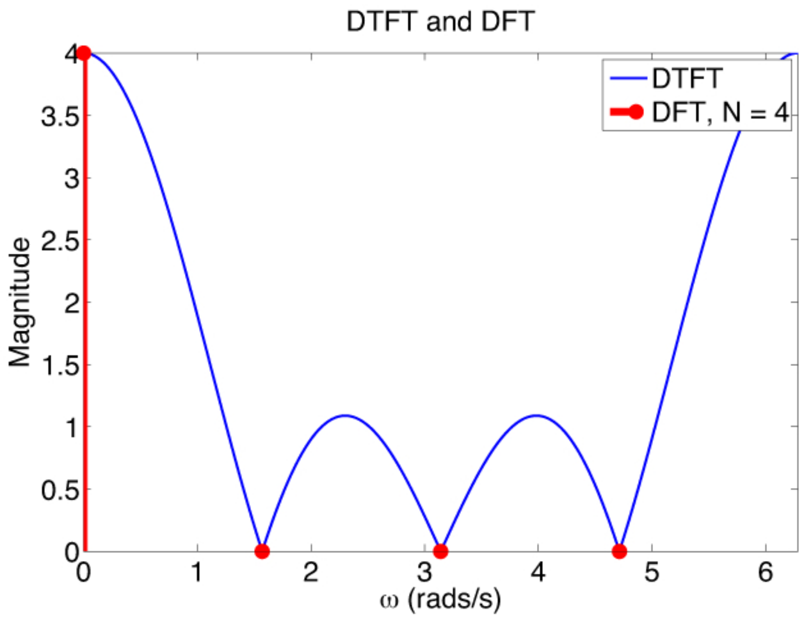
$$= \begin{cases} 4 & k=0 \\ 0 & k=1,2,3 \end{cases}$$

DFT

$$X[k] = \sum_{n=0}^{3} 1\cdot e^{-j\frac{2\pi}{4}kn}$$

$$= 1 + e^{-j\pi/2\,k} + e^{-j\pi k} + e^{-j\frac{3\pi}{2}k}$$

$$= \begin{cases} 4 & k=0 \\ 0 & k=1,2,3 \end{cases}$$



Sampling in frequency?

- recall sampling in time creates periodicity (replicates) in freq.

Sample $X(e^{j\omega})$ at $\omega_k = k\frac{2\pi}{M}$, $k = 0, 1, 2, \cdots M-1$

$$X_s[k] = X(e^{jk2\pi/M}) \approx X(e^{j\omega})\, S(e^{j\omega})$$

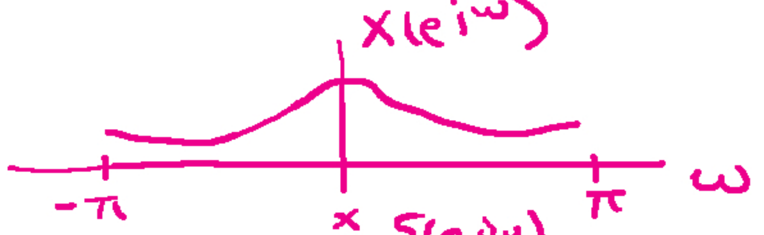
$$\Big\downarrow \text{DTFT}$$

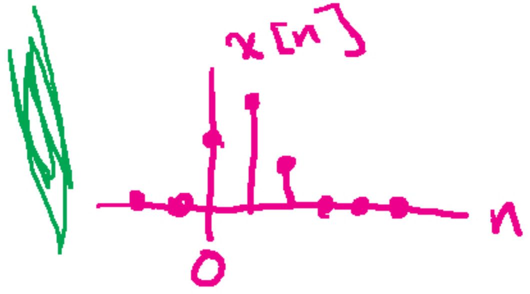
$$x_s[n] \xleftarrow{\text{DFT}} X_s[k]$$

$$x_s[n] = x[n] * s[n]$$

$$= x[n] * \sum_{\ell=-\infty}^{\infty} \delta[n - \ell M]$$

$$= \sum_{\ell=-\infty}^{\infty} x[n - \ell M]$$

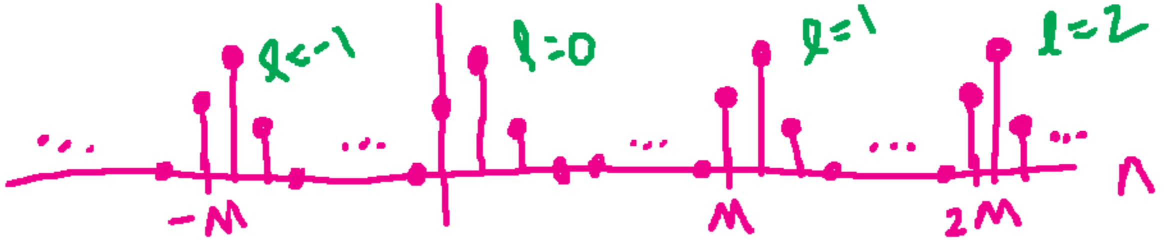




Sample DTFT $\longrightarrow$

$\omega_k = \frac{2\pi k}{M}$

$x_s[n] = x[((n))_M]$



Sampling in frequency ⇒ replication in time

(Sampling in time ⇒ replication in frequency)

If $x[n] \xleftrightarrow{\text{DTFT}} X(e^{j\omega})$ and $X_s[k] = X(e^{j\omega})\Big|_{\omega=\frac{2\pi}{M}k,\ k=0,1,\cdots M-1}$

then $X_s[k] \xleftrightarrow{\text{DFT}} x_s[n] = \sum_{\ell=-\infty}^{\infty} x[n-\ell M]$

To "recover" $x[n]$ from $x_s[n]$, $x[n]$ must be time-limited to $\leq M$ values (otherwise aliasing)

Use of M-point DFT ⇒ time signal is M-periodic