Reconstruction and The Sampling Theorem

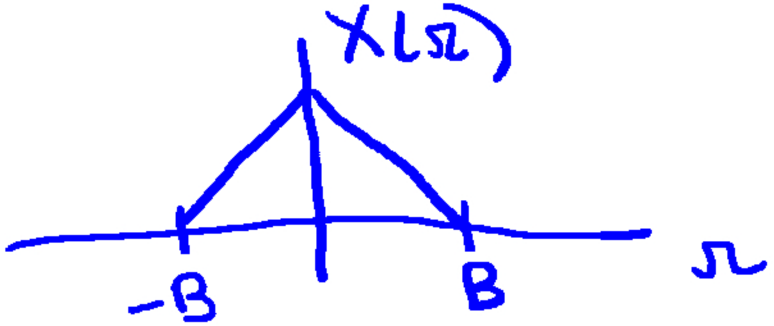
Recall - If $x(t) \xleftrightarrow{FT} X(\Omega)$ and $x[n] = x(nT)$, then

$$x[n] \xleftrightarrow{FT} X_s(\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(\Omega - k\Omega_s).$$

$\Omega_s = \frac{2\pi}{T}$: Sampling frequency $\left(\frac{1}{T} Hz\right)$

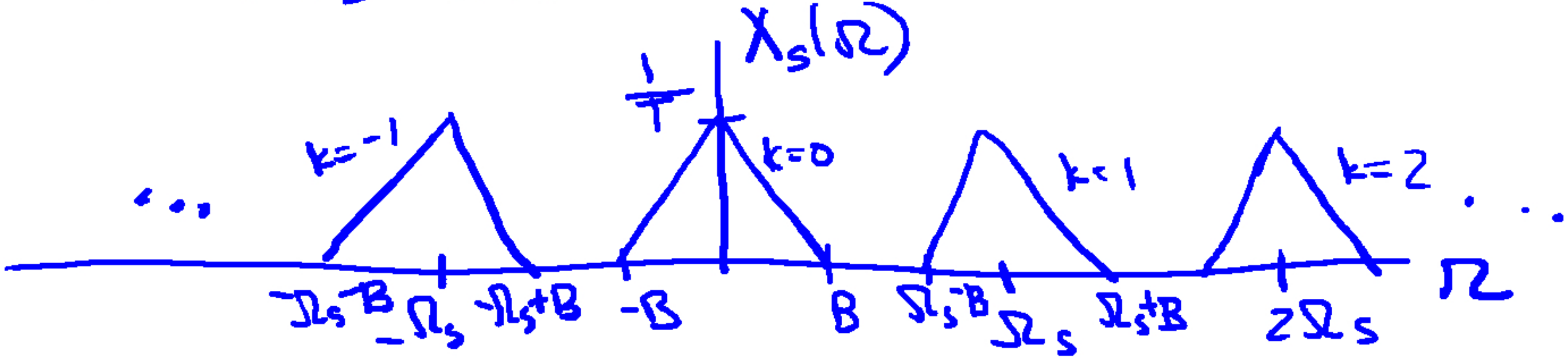
Reconstruction: find $x(t)$ given $x[n]$

Assume $x(t)$ is bandlimited



$X(\Omega) = 0, \; |\Omega| > B$

① Consider $\Omega_s > 2B$

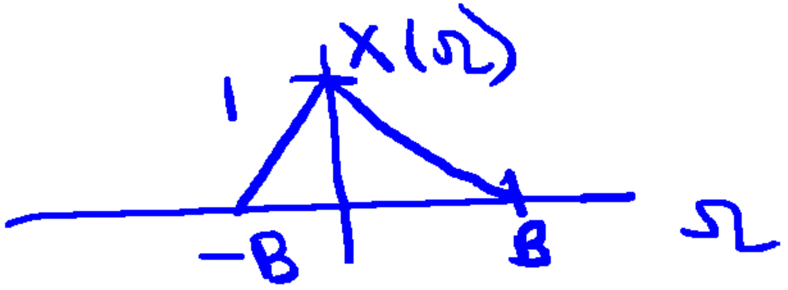


Get $X(\Omega)$ from $X_s(\Omega)$

$\times$

low pass filter

$=$

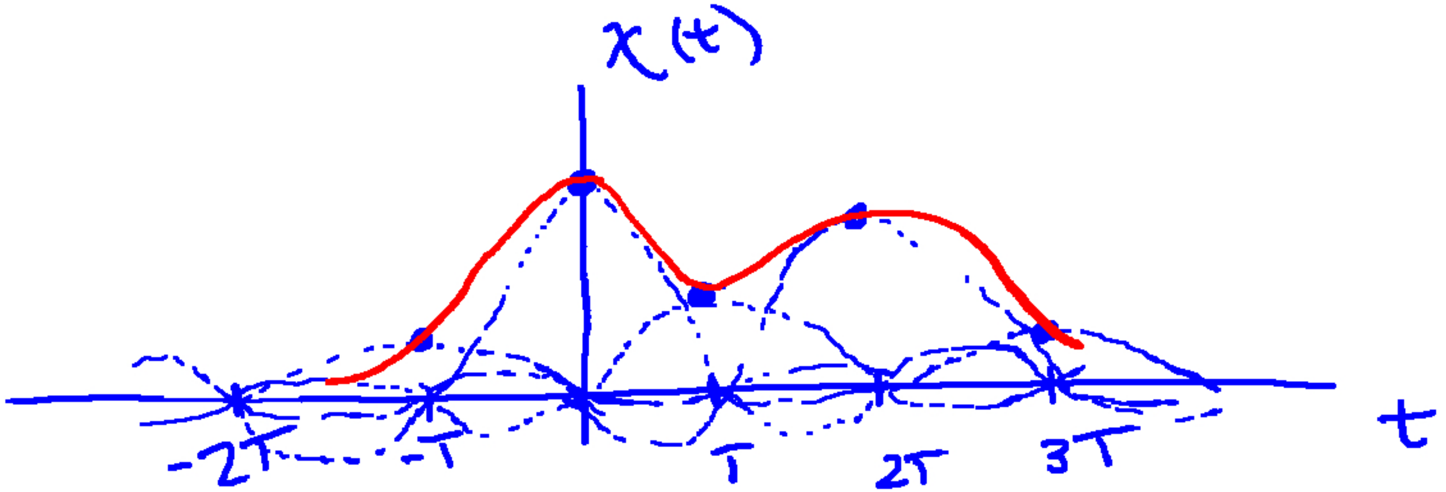


$$X(\Omega) = X_s(\Omega) H_r(\Omega) \xleftrightarrow{FT} x(t) = \sum_{n=-\infty}^{\infty} x[n]\delta(t-nT) * h_r(t)$$

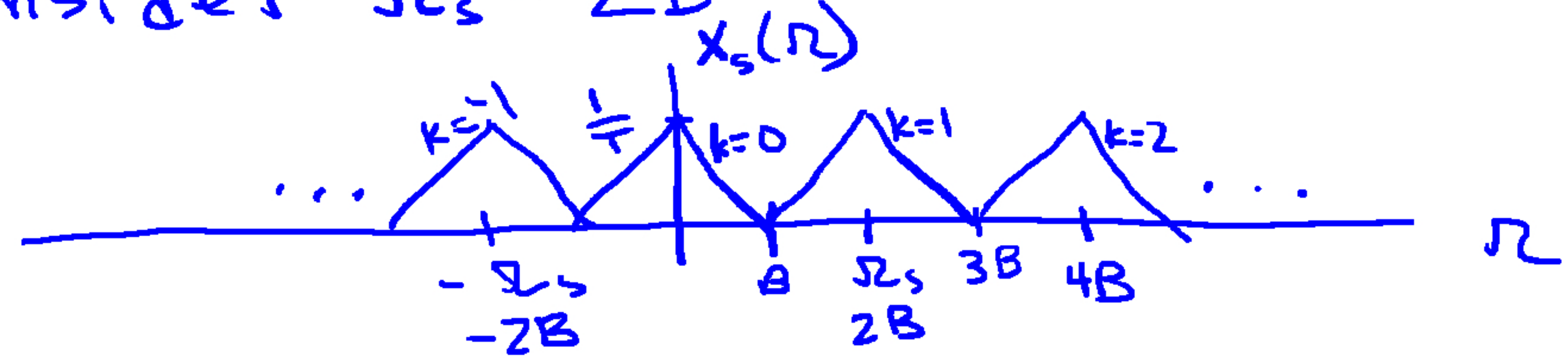
$$x(t) = \sum_{n=-\infty}^{\infty} x[n] h_r(t-nT) \qquad h_r(t) = T\,\frac{\sin\left(\frac{\Omega_s}{2} t\right)}{\pi t}$$

$$x(t) = T\sum_{n=-\infty}^{\infty} x[n] \frac{\sin\left(\frac{\Omega_s}{2}(t-nT)\right)}{\pi(t-nT)}$$

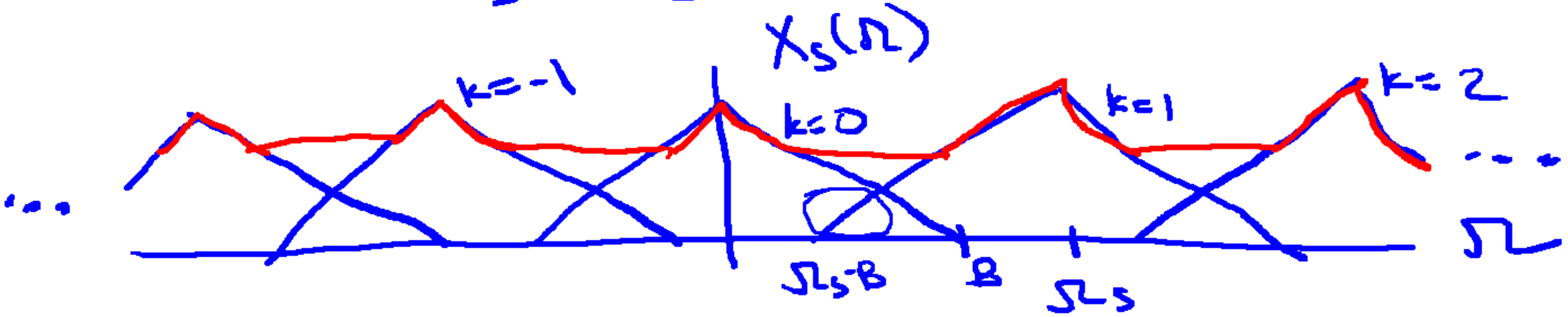
ideal bandlimited interpolation



② Consider $\Omega_s = 2B$



③ Consider $\Omega_s < 2B$



overlap – cannot uniquely recover $x(t)$ from $x[n]$

Aliasing – frequency comps. in $X(\Omega - \Omega_s)$ show up in the "wrong" place

Sampling Theorem (Nyquist)

Let $x(t)$ be bandlimited with $X(\Omega)=0$ for $|\Omega|>B$

Then $x(t)$ is uniquely determined by its samples $x[n]=x(nT)$ provided

$$\Omega_s = \frac{2\pi}{T} > 2B$$

$\Omega = B$: Nyquist frequency

$\Omega = 2B$: Nyquist rate