

# Theoretical uncertainties for electroweak and Higgs-boson precision measurements at FCC-ee

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## Abstract

Due to the high anticipated experimental precision at the Future Circular Collider FCC-ee (or other proposed  $e^+e^-$  colliders, such as ILC, CLIC, or CEPC) for electroweak and Higgs-boson precision measurements, theoretical uncertainties may have, if unattended, an important impact on the interpretation of these measurements within the Standard Model (SM), and thus on constraints on new physics. Current theory uncertainties, which would dominate the total uncertainty, need to be strongly reduced through future advances in the calculation of multi-loop radiative corrections together with improved experimental and theoretical control of the precision of SM input parameters. This document aims to provide an estimate of the required improvement in calculational accuracy in view of the anticipated high precision at the FCC-ee. For the most relevant electroweak and Higgs-boson precision observables we evaluate the corresponding quantitative impact.

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# 1 Introduction

With the discovery of the Higgs boson, all elements of the Standard Model have been experimentally confirmed and tested in great depth. On the other hand, observational evidence for neutrino masses, dark matter and the matter-antimatter asymmetry require physics beyond the Standard Model. One promising way to probe such new physics is through precision measurements of the properties of the electroweak gauge bosons and the Higgs boson. This is the avenue pursued by several proposals for a future  $e^+e^-$  collider. In particular, the FCC-ee concept is designed to run at the  $Z$  pole, the  $WW$  threshold, as a Higgs factory, and at the  $t\bar{t}$  threshold, and thus it can improve indirect probes for new physics from all of these by several orders of magnitude compared to existing bounds [1, 2].

The anticipated experimental accuracy of an observable has to be matched with a theory prediction of at least the same level of accuracy to make maximum use of the experimental data. Both types of uncertainties (theoretical and experimental) must be taken into account when deriving constraints on new physics from the data.

Several sources of theory uncertainties have to be distinguished. The intrinsic uncertainties are due to missing higher-orders in the perturbative expansion of the SM (or BSM) prediction for an observable. The parametric uncertainties are due to the imperfect experimental knowledge of the SM input parameters as well as theory uncertainties induced in their extraction from data<sup>1</sup>. The extraction of a quantity from a cross section or an asymmetry requires the theory prediction of this cross section or asymmetry to at least the same order of precision<sup>2</sup>. Finally, it is worth mentioning that the achievable precision for the prediction of a given observable is not only limited by the theory uncertainty to this observable. In many cases, the determination of the “observable” itself from experimental data requires theory input for the subtraction of background and/or the evaluation of the impact of experimental acceptances. Thus the typical electroweak precision observables and Higgs precision observables are technically “pseudo-observables”.

In this paper the current status and future implications of all three types of theory uncertainties will be summarized. While they have been obtained for the FCC-ee, they are valid for all future high precision  $e^+e^-$  colliders running on the  $Z$  pole or above the  $HZ$  threshold, such as ILC, CLIC, or CEPC. We will use anticipated FCC-ee precisions to illustrate the impact of theory uncertainties. This compilation may serve as a reference for other FCC-ee (or similar) studies.

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<sup>1</sup>The distinction between observables and input parameters is somewhat arbitrary in global electroweak fits. However, if an observable is affected by a large uncertainty from an input parameter, the precision of the global fit will typically be impacted to a similar degree. For this reason we will refrain from discussing global fits separately in this document.

<sup>2</sup>It is worth noting that in some cases the impact of theory uncertainties can be reduced by analyzing certain ratios or difference of measured quantities.

## 2 Methods for estimation of theory errors

Some of the commonly used techniques for the determination of theory errors are (for a review see Ref. [3])

- Determine relevant prefactors pertaining to a class of higher-order corrections, such as couplings, group factors, particle multiplicities, mass ratios, *etc.* It is assumed that the remainder of the loop amplitude is  $\mathcal{O}(1)$ .
- If several orders of radiative corrections to some quantity have already been computed, one can extrapolate to higher orders based on the observation that the perturbation series approximately follows a geometric series.<sup>3</sup>
- When using  $\overline{\text{MS}}$  renormalization, one can use the scale dependence of a given fixed-order result to estimate the size of the missing higher orders. This method is commonly used in the context of QCD corrections, but is less useful for electroweak calculations.
- Compare the results in different renormalization schemes, such as the on-shell and  $\overline{\text{MS}}$  schemes, where the differences are of the next order in the perturbative expansion.

There is no formal argument concerning the validity of any of these methods and their usefulness is mostly supported based on experience. The estimates in the following subsections are mostly derived using the prefactor and geometric series methods. The reader should take the numbers presented below with a grain of salt and be aware that they are educated guesses rather than robust quantitative predictions.

## 3 SM parameter determination

The prediction of precision (pseudo-)observables depends on the input quantities  $G_F$ ,  $\alpha(M_Z) \equiv \alpha/(1 - \Delta\alpha)$ ,  $\alpha_s(M_Z)$ ,  $M_Z$ ,  $M_H$ ,  $m_t$  and  $m_b$  (and  $m_c$ ). These need to be determined with sufficient precision from independent measurements.  $G_F$  is currently known with a very small uncertainty of 0.5 ppm, such that it will be a subdominant uncertainty source for future FCC-ee precision tests. All of the remaining parameters can, in principle, be measured at FCC-ee with improved precision compared to current world averages. However, each of these measurements is also subject to theory uncertainties itself. Here we list the future expectations for the most relevant SM input parameters, as derived for the FCC-ee. For other  $e^+e^-$  collider experiments different uncertainties are expected, and the relevant numbers given below could be rescaled.

$M_H$ : The Higgs mass can be measured from  $e^+e^- \rightarrow HZ$  at  $\sqrt{s} = 240$  GeV with a precision of the order of 10 MeV [4, 5]. At this level of precision, theory uncertainties (mainly due to final-state radiation effects) are subdominant.

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<sup>3</sup>It should be kept in mind that the series coefficients receive combinatorial enhancement factors at higher orders. The method described here is a heuristic ansatz that empirically works for extrapolations to two- or three-loop order.

$M_Z$ : The  $Z$  mass can be determined from the line-shape of the process  $e^+e^- \rightarrow f\bar{f}$  at several center-of-mass energies above and below  $\sqrt{s} = M_Z$ . The experimental precision at FCC-ee is estimated to be below 0.1 MeV. On the theory side, this measurement is primarily affected by QED initial-state radiation and interference between initial and final-state. Smaller modifications arise from non-resonant photon exchange and electroweak box diagrams. In Ref. [6] it was estimated that the uncertainty for the current best calculations of these effects amounts to less than 0.1 MeV.

$\alpha_s(M_Z)$ : Given the overconstrained system of electroweak precision pseudo-observables, the value of  $\alpha_s(M_Z)$  can be extracted from a global fit to these quantities. The experimental uncertainty from such a fit to FCC-ee data is projected to be about  $10^{-4}$  [7]. One advantage of this approach is that the strong coupling is obtained directly at the scale  $\mu = M_Z$ , where non-perturbative uncertainties are suppressed by powers of  $M_Z^2$ , in contrast to  $\alpha_s$  determinations from hadronic  $\tau$  decays or jet event shapes. However, perturbative uncertainties will be important. To evaluate their impact, one can look at  $R_\ell$ , which is one of the most sensitive pseudo-observables for  $\alpha_s$ . With  $\delta_{\text{th}}R_\ell = 1.5 \times 10^{-3}$ , see Tab. 1 below, one obtains a theory uncertainty of 0.00015 for  $\alpha_s(M_Z)$  in addition to the experimental uncertainty.

$m_t$  (defined as the  $\overline{\text{MS}}$  mass,  $m_t(m_t)$ ): The top-quark mass can be determined from a threshold scan near the pair production threshold at  $\sqrt{s} \approx 350$  GeV, using a threshold mass definition (1S or PS). The projected experimental uncertainty for this measurement is  $\delta m_t \approx 17$  MeV [2]. On the theory side, there are several uncertainty sources [9]: (i) The perturbative uncertainty for the calculation of the threshold shape. The currently most precise result includes NNNLO QCD [10] and NNLO EW and non-resonant corrections (counting orders in non-relativistic, resummed perturbation theory for the top threshold) [11], resulting in an error of  $m_t$  of somewhat less than 50 MeV. It is expected that this can be further improved by matching the fixed-order calculation with resummed calculations based on effective field theories, currently available at the NNLL level [13], but it may be difficult to reduce the error by a factor of two or more. (ii) The threshold shape calculations are performed by using a threshold mass definition (1S or PS), which must be translated into the  $\overline{\text{MS}}$  scheme. This translation is currently known to the four-loop level [14], resulting in an uncertainty of about 10 MeV. (iii) The threshold shape prediction and mass scheme translation depend on the value of  $\alpha_s$ . Assuming an uncertainty  $\delta\alpha_s \sim 0.001$  leads to a top-quark mass error of  $\delta m_t \sim 15$  MeV for the measurement of a top threshold mass definition [11], but to an uncertainty in the conversion to the  $\overline{\text{MS}}$  mass of roughly 70 MeV [12]. With a future improved measurement of  $\alpha_s$  at FCC-ee (see above), the latter error contribution can also be reduced to the level of 15 MeV. Combining these three error sources, a theory uncertainty of less than 50 MeV for  $m_t$  appears feasible.

Besides the issues discussed above, the achievement of this goal will require a very accurate determination of the efficiencies of experimental acceptances and selection cuts. This task is facilitated by the inclusion of higher-order corrections and resummation results in a Monte-Carlo event generator. Work in this direction is underway, with recent evaluations of the NNLL resummed threshold-shape in the presence of phase-space cuts [13], NLO QCD corrections for off-shell  $t\bar{t}$  production [15], and matching between these contributions [16], complementing the semi-analytic approach [17, 18] implemented in the most recent NNNLO

QCD + NNLO EW/non-resonant calculation [11].

$\underline{m_b}$  and  $\underline{m_c}$  (defined as  $\overline{\text{MS}}$  masses,  $m_b(m_b)$  and  $m_c(m_c)$ ): We use estimated future values of  $\delta m_b(m_b) \sim 13$  MeV and  $\delta m_c(m_c) \sim 7$  MeV [19]. These are based on projected improvements in lattice calculations, for which we take the moderately conservative LS scenario in Ref. [19]. Note that some analysis based on QCD sum rules already claim an uncertainty of  $\delta m_b \sim 10$  MeV [20], but these error estimates are not confirmed in other analyses of the same quantities [21,22]. It would be very welcome to have the two independent results from lattice and QCD sum rules for cross-checking and for putting uncertainty estimates to a more solid basis.

$\underline{\Delta\alpha}$ : The potentially most difficult parameter to measure is the shift in the electromagnetic fine structure constant,  $\Delta\alpha \equiv 1 - \alpha(0)/\alpha(M_Z)$ . It is traditionally determined from data on  $e^+e^- \rightarrow$  hadrons and tau decays to hadrons [23], with a current uncertainty of  $\delta(\Delta\alpha) = \mathcal{O}(10^{-4})$ . More accurate data for these inputs is expected to become available from BES III [24], VEPP-2000 [25] and Belle II [26]. It was estimated that an uncertainty of  $4 \times 10^{-5}$  to  $5 \times 10^{-5}$  could be reached, depending on improvements in QCD theory input [27]. At FCC-ee, independent measurements of  $e^+e^- \rightarrow$  hadrons could be obtained through the radiative return method [28]. The ultimate precision on  $\Delta\alpha$  that could be reached by this method is not yet clear, but in view of the importance of the quantity  $\Delta\alpha$  for EW processes it would be highly desirable to have independent results on  $\Delta\alpha$  for a firm uncertainty assessment.

In addition, the expected high luminosity may, for the first time, enable the possibility to measure  $\alpha(M_Z)$  directly from  $e^+e^- \rightarrow f\bar{f}$  at  $\sqrt{s_{\pm}} = M_Z \pm 3$  GeV [29]. In Ref. [29] it was evaluated that from an analysis of the FB asymmetry of  $e^+e^- \rightarrow \mu^+\mu^-$  one can determine  $\alpha(M_Z)$  with an experimental uncertainty of  $3 \times 10^{-5}$ . However, this also requires theory input for the subtraction of other electroweak corrections from the pure  $s$ -channel photon exchange contribution, which is the part that directly depends on  $\alpha(M_Z)$ . In particular, one needs to subtract contributions from  $s$ -channel  $Z$  exchange, box diagrams, and corrections to the  $\gamma ff$  vertex. All of these contributions are currently at the one-loop level (see for example Ref. [30]) and have an impact of  $\mathcal{O}(10^{-3})$  on the extracted value of  $\alpha(M_Z)$ . This value is relatively small due to partial cancellations between  $A_{\text{FB}}$  at  $s_+$  and  $s_-$ , but still large compared to the experimental target. With existing loop calculation methods, it is possible to compute complete fermionic two-loop corrections, as well as  $\mathcal{O}(\alpha\alpha_s^2)$  corrections. After inclusion of these contribution, the remaining theory uncertainty is estimated to amount to  $\mathcal{O}(10^{-4})$ . If additionally the full  $\mathcal{O}(\alpha^2)$ ,  $\mathcal{O}(\alpha^2\alpha_s)$  and double-fermionic  $\mathcal{O}(\alpha^3)$  corrections become available, this uncertainty may be reduced to the level of a few times  $10^{-5}$ . These calculations (which include two- and three-loop box diagrams) would require new developments for loop integration techniques, but may be achievable in the future.

There are also important QED effects in  $A_{\text{FB}}(s_+) - A_{\text{FB}}(s_-)$  due to initial-final state interference, examined in the recent study of Ref. [31]. They are numerically much bigger than non-QED EW corrections – amounting to about 0.5%, in spite of the partial cancellation in the difference. It was shown in Ref. [31], that thanks to an advanced technique for soft photon resummation, it can be controlled theoretically with a precision of  $\mathcal{O}(0.01\%)$ . Another factor  $\sim 10$  improvement is needed in order to match FCC-ee experimental precision

of the direct measurement of  $\alpha(M_Z)$ .

For illustration, two scenarios for  $\Delta\alpha$  will be considered below, one with total anticipated uncertainty of  $5 \times 10^{-5}$  (assuming a combination of the experimental and possible future theory uncertainties of similar magnitude) and an optimistic one with total uncertainty of  $3 \times 10^{-5}$  (corresponding to subdominant theory uncertainties). For the optimistic scenario, we also consider a reduced uncertainty of  $\alpha_s$ , which may be achievable by combining several observables [32].

Taking into account the experimental and theoretical uncertainties discussed above, one arrives at the following estimates for the achievable precision (from direct determination) for the most important SM parameters at the FCC-ee:

$$\begin{aligned} \delta m_t &= 50 \text{ MeV}, & \delta m_b &= 13 \text{ MeV}, & \delta M_Z &= 0.1 \text{ MeV}, & \delta \alpha_s &= 0.0002 \text{ (0.0001)}, \\ \delta(\Delta\alpha) &= 5 \times 10^{-5} \text{ (} 3 \times 10^{-5}\text{)}. \end{aligned} \quad (1)$$

For  $m_t$  and  $\alpha_s$ , another factor two improvement could be envisioned with a more ambitious theory advancement.

## 4 Electroweak precision observables

### 4.1 EWPO definitions

The most important electroweak precision observables (EWPO) are related to properties of the  $Z$  and  $W$  bosons.  $Z$ -boson properties are determined from measurements of  $e^+e^- \rightarrow f\bar{f}$  on the  $Z$ -pole. To isolate the physics of the  $Z$ -boson, the typical set of pseudo-observables is defined in the terms of the de-convoluted cross-section  $\sigma_f(s)$ , where the effect of initial- and final-state photon radiation and from  $s$ -channel photon and double-boson (box) exchange has been removed. The impact of these corrections will be discussed below. The customary set of pseudo-observables are

$$\sigma_{\text{had}}^0 = \sum_q \sigma_q(M_Z^2), \quad (2)$$

$$\Gamma_Z = \sum_f \Gamma[Z \rightarrow f\bar{f}], \quad (\text{from a fit to } \sigma_f(s) \text{ at various values of } s) \quad (3)$$

$$R_\ell = [\sum_q \sigma_q(M_Z^2)] / \sigma_\ell(M_Z^2), \quad (\ell = e, \mu, \tau) \quad (4)$$

$$R_q = \sigma_q(M_Z^2) / [\sum_q \sigma_q(M_Z^2)], \quad (q = b, c) \quad (5)$$

$$A_{\text{FB}}^f = \frac{\sigma_f(\theta < \frac{\pi}{2}) - \sigma_f(\theta > \frac{\pi}{2})}{\sigma_f(\theta < \frac{\pi}{2}) + \sigma_f(\theta > \frac{\pi}{2})} \equiv \frac{3}{4} \mathcal{A}_e \mathcal{A}_f, \quad (6)$$

$$A_{\text{LR}}^f = \frac{\sigma_f(P_e < 0) - \sigma_f(P_e > 0)}{\sigma_f(P_e < 0) + \sigma_f(P_e > 0)} \equiv \mathcal{A}_e |P_e|. \quad (7)$$

Quantity	FCC-ee	Current intrinsic error	Projected intrinsic error
$M_W$ [MeV]	0.5–1 <sup>‡</sup>	4 ( $\alpha^3, \alpha^2\alpha_s$ )	1
$\sin^2 \theta_{\text{eff}}^\ell$ [ $10^{-5}$ ]	0.6	4.5 ( $\alpha^3, \alpha^2\alpha_s$ )	1.5
$\Gamma_Z$ [MeV]	0.1	0.4 ( $\alpha^3, \alpha^2\alpha_s, \alpha\alpha_s^2$ )	0.15
$R_b$ [ $10^{-5}$ ]	6	11 ( $\alpha^3, \alpha^2\alpha_s$ )	5
$R_l$ [ $10^{-3}$ ]	1	6 ( $\alpha^3, \alpha^2\alpha_s$ )	1.5

<sup>‡</sup>The pure experimental precision on  $M_W$  is  $\sim 0.5$  MeV [1, 2], see Sec. 4.2.2 for more details.

Table 1: Estimated precision for the direct determination of several important electroweak precision observables at FCC-ee [1, 2, 33] (column two, including systematic and observable-specific) uncertainties; as well as current intrinsic theory errors for the prediction of these quantities within the SM (column three). The main sources of theory errors are also indicated. Column four shows the estimated projected intrinsic theory errors when leading 3-loop corrections become available. See text for more details.

Here  $\theta$  is the scattering angle and  $P_e$  is the polarization of the incoming electron beam.<sup>4</sup> The asymmetry parameters are commonly written as

$$\mathcal{A}_f = \frac{1 - 4|Q_f| \sin^2 \theta_{\text{eff}}^f}{1 - 4|Q_f| \sin^2 \theta_{\text{eff}}^f + 8(Q_f \sin^2 \theta_{\text{eff}}^f)^2}. \quad (8)$$

Here  $Q_f$  denotes the charge of the fermion, and  $\sin^2 \theta_{\text{eff}}^f$  is the effective weak (fermionic) mixing angle. Another important precision observable is the  $W$ -boson mass. It is currently measured most precisely from the lepton  $p_\perp$  distribution in  $pp \rightarrow \ell\nu$  at hadron colliders, and it can be calculated within the SM from the Fermi constant,  $G_F$ , of muon decay.

The expected precision for the experimental determination of some of these quantities at FCC-ee is given in the second column of Tab. 1 [1, 2, 33]. The  $Z$ -boson quantities can be determined from a run at  $\sqrt{s} = M_Z$  with several  $\text{ab}^{-1}$ , and smaller statistics runs at center-of-mass energies above and below the  $Z$  peak for the purpose of  $M_Z$  and  $\Gamma_Z$  measurements. The  $W$  mass can be determined from a run at several values of  $\sqrt{s}$  near the threshold  $2M_W$  with a combined luminosity of  $\mathcal{O}(\text{ab}^{-1})$ . Note that the number for  $M_W$  in the table includes an estimate of the theory error as described in section 4.2.2, since the measurement of  $M_W$  requires a full SM prediction (not only QED) for the  $WW$  cross-section near threshold as input.

## 4.2 Theory uncertainties for EWPO

### 4.2.1 Intrinsic uncertainties

The quantities listed in Tab. 1 can be predicted within the SM by using  $G_F$ ,  $\alpha(M_Z)$ ,  $\alpha_s(M_Z)$ ,  $M_Z$ ,  $M_H$  and  $m_t$  as inputs. The radiative corrections in these predictions are currently

<sup>4</sup>Formulas for electron and positron polarization can be found, e.g., in Ref. [8].

known including complete two-loop corrections. In addition, approximate three- and four-loop corrections of  $\mathcal{O}(\alpha_t^3)$ ,  $\mathcal{O}(\alpha_t^2\alpha_s)$ ,  $\mathcal{O}(\alpha_t\alpha_s^2)$  and  $\mathcal{O}(\alpha_t\alpha_s^3)$  are available, where  $\alpha_t = y_t^2/(4\pi)$  and  $y_t$  is the top Yukawa coupling. For a review, see Ref. [3]. The theory uncertainties from missing higher-order corrections are given in the third column of Tab. 1 [34]. Also indicated are the main sources for the respective uncertainties.

As evident from the table, to match the anticipated FCC-ee precision, substantial improvements in the SM theory prediction are necessary. In Ref. [35,36], it was estimated how the intrinsic uncertainty will likely be reduced if the following calculations become available: complete  $\mathcal{O}(\alpha\alpha_s^2)$  corrections, fermionic  $\mathcal{O}(\alpha^2\alpha_s)$  corrections, double-fermionic  $\mathcal{O}(\alpha^3)$  corrections, and leading four-loop corrections enhanced by the top Yukawa coupling.

With the inclusion of these corrections, the estimated future intrinsic uncertainties will become comparable to the anticipated experimental FCC-ee precision, as shown in the fourth column of Tab. 1. However, for some quantities the calculation of additional terms will be necessary to match the anticipated experimental accuracies, in particular for  $\sin^2\theta_{\text{eff}}^\ell$ . These extra terms include subdominant three-loop and leading four-loop contributions<sup>5</sup>.

To carry out these calculations, qualitatively new developments of existing loop integration techniques will be required, but no conceptual paradigm shift. An extensive overview of recent progress in loop calculation techniques and prospects for future developments can be found in Ref. [36,37]. For the calculation of full SM corrections, numerical integration techniques are often advantageous due to the large number of independent mass and momentum scales involved. Some techniques, such as sector decomposition and Mellin-Barnes representations can in principle be applied to problems with arbitrary number of legs and loops. Several public computer codes based on these methods are available. Nevertheless, substantial improvements will be needed to ensure reliable numerical precision of at least 2–3 digits for future 3-loop and 4-loop applications. In some cases where higher precision is needed, more specialized semi-numerical methods can be applied. On the other hand, for QED and QCD corrections on external legs (see below), analytical techniques are typically more efficient if the radiation is treated inclusively, or Monte-Carlo techniques if experimental cuts are applied to the phase-space. See Ref. [36,37] for more details.

On the other hand, it should be kept in mind that the determination of the pseudo-observables in Tab. 1 from experimental data also requires theory input for the removal of initial-state and final-state photon radiation and  $s$ -channel photon exchange and box contributions.

For the total cross-section for  $e^+e^- \rightarrow f\bar{f}$ , the contributions from QED radiation are currently known with complete  $\mathcal{O}(\alpha)$  and  $\mathcal{O}(\alpha^2)$  corrections and log-enhanced  $\mathcal{O}(\alpha^3L^3)$  corrections. For the asymmetries,  $\mathcal{O}(\alpha)$  and  $\mathcal{O}(\alpha^2L^2)$  corrections are available (with  $L \equiv \log s/m_e^2$ ). More details can be found, for example, in Ref. [38], see also the discussion in Ref. [39].

The theory uncertainty from missing higher QED orders is estimated to amount to a few times 0.01% [38,40] for the  $Z$ -peak cross-section and total width measurements. In order not to be dominated by this uncertainty, it will need to be reduced by about a factor 10 for the FCC-ee. This will require the calculation of non-leading log contributions to

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<sup>5</sup>This future scenario is called TH2 in Ref. [36].



$\mathcal{O}(\alpha^3)$  corrections,  $\mathcal{O}(\alpha^3 L^2)$  and  $\mathcal{O}(\alpha^4 L^4)$  contributions, as well as an improved treatment of fermion pair production from off-shell photons.

Generally, the concept of pseudo-observables to describe the  $Z$  line shape and asymmetries should be validated (and modified if necessary) at the aimed target precision. On the one hand, a full calculation of the resonance process at  $\mathcal{O}(\alpha^2)$ , including all off-shell contributions and weak corrections, seems necessary. Technically, this is a challenging two-loop calculation with massive particles; conceptually, a method to treat resonance processes (including off-shell tails) is required at the two-loop level. On the other hand, the field-theoretical definition of the pseudo-observable should be consistently based on the complex  $Z$  pole, and the parametrization of the relevant cross sections via pseudo-observables should be carefully compared to the best achievable prediction. Though very challenging, we are confident that both steps can be made in theory.

The role of theory uncertainties in the determination of  $M_W$  from a threshold scan will be discussed in the next subsection.

#### 4.2.2 Parametric uncertainties

As discussed above, the prediction of electroweak precision pseudo-observables depends on the input quantities  $G_F$ ,  $\alpha(M_Z) \equiv \alpha/(1 - \Delta\alpha)$ ,  $\alpha_s(M_Z)$ ,  $M_Z$ ,  $M_H$  and  $m_t$ . Their anticipated precisions, including future FCC-ee measurements are summarized in Eq. 1. In the following we discuss the impact of each of these parameters on the EWPO.

$M_H$ : The Higgs mass can be measured with a precision of the order of 10 MeV. The dependence of other electroweak precision pseudo-observables on  $M_H$  is proportional to  $\alpha \log M_H/M_W$  and thus relatively mild, so that an accuracy below 0.05 GeV will lead to a negligible error contribution for electroweak precision tests.

$M_Z$ : The  $Z$  mass is expected to be determined to better than 0.1 MeV. This very small error will not be a limiting factor for electroweak precision fits.

$\alpha_s(M_Z)$ : With a theory uncertainty of 0.00015 for  $\alpha_s(M_Z)$  in addition to the experimental uncertainty, one arrives at the parametric uncertainties as given in Tab. 2.

$m_t$ : With a top quark mass uncertainty of 50 MeV one arrives at the parametric uncertainties as given in Tab. 2.

$m_b$ : The impact of the bottom-quark mass cannot be entirely neglected, but its uncertainty contribution is negligibly small.

$\Delta\alpha$ : For illustration, we use two scenarios for  $\Delta\alpha$  in our evaluations, one with total anticipated uncertainty of  $5 \times 10^{-5}$  (assuming a combination of the experimental and possible future theory uncertainties of similar magnitude) and an optimistic one with total uncertainty of  $3 \times 10^{-5}$  (corresponding to subdominant theory uncertainties). The corresponding results can be found in Tab. 2.

$M_W$ : Here we also discuss the  $W$ -boson mass, which will (most likely) not serve as an input parameter. Similarly to the top quark mass,  $M_W$  can be determined from a threshold scan near the  $W$ -pair threshold,  $\sqrt{s} \approx 161$  GeV. It is foreseen that the experimental uncertainty at FCC-ee for this measurement is about 0.5 MeV [1,2]. At the point of highest sensitivity,

an uncertainty of the cross-section measurement of 0.1% translates to an uncertainty of  $\sim 1.5$  MeV on  $M_W$  [41]. Therefore a theoretical prediction for the process  $e^+e^- \rightarrow 4f$  with an accuracy of  $\Delta\sigma \sim 0.01\%$  is desirable, including effects of off-shell W bosons, which become important near threshold.

The currently best calculations are based on complete one-loop results for  $e^+e^- \rightarrow 4f$  [42] and partial higher-order effects for the total cross section from an effective field theory framework [43,44]. The resulting theory uncertainty on  $M_W$  is estimated to be about 3 MeV [44]. Building on the effective field theory framework of Ref. [43], this result could be improved if complete 2-loop corrections to  $e^+e^- \rightarrow W^+W^-$  and to  $W \rightarrow f\bar{f}'$  become available (which are required for the determination of matching coefficients in the effective theory). While such a calculation poses a challenge, it may be feasible with a serious effort over some number of years. In addition, a more accurate description of initial-state radiation will be important, which includes universal contributions from soft and collinear photon radiation (see Ref. [38] for a review), as well as hard photon radiation. For the latter, a proper matching and merging procedure needs to be employed to avoid double counting [43,45]. Estimates based on scaling arguments [46] or explicit calculation of a subset of leading, Coulomb enhanced, three-loop effects [47] suggest that corrections beyond NNLO are of the order of 0.02%. The full computation of the Coulomb-enhanced three-loop corrections is expected to be feasible after completion of the NNLO EFT calculation. The remaining, non Coulomb-enhanced  $\mathcal{O}(\alpha^3)$  corrections are therefore expected to be below the FCC-ee target accuracy. The magnitude of single- or non-resonant contributions have been estimated as 0.016-0.03% [46,47]. These corrections would be included in a full NNLO computation of  $e^+e^- \rightarrow 4f$ , but it may also be possible to compute them within the EFT. These estimates suggest a theory induced systematic error  $\Delta M_W = (0.15 - 0.60)$  MeV, where the lower value results from assuming the non-resonant corrections are under control.

Current cross-section predictions for  $e^+e^- \rightarrow WW \rightarrow 4f$  involving final-state quarks do not yet fully control all effects of  $\mathcal{O}(\alpha_s^2)$ , which is, however, needed to meet the 0.01%–0.04% precision tag for the total cross section. For QCD corrections of “factorizable” origin, which correspond to corrections to hadronic on-shell or off-shell  $W/Z$  decays, these corrections are straightforward to supplement, at least for integrated quantities. “Non-factorizable” QCD interconnection corrections, which involve two-parton exchange between two hadronically decaying  $W$  or  $Z$  bosons, are more complicated. Their leading resonant contribution should be calculable, so that the remaining uncertainty on those effects will be  $\mathcal{O}(\alpha_s^2/\pi^2 \times \Gamma_W/M_W) < 0.01\%$ . It is even expected [48] that the resonant part of the non-factorizable QCD interconnection effects vanishes to all orders in  $\alpha_s$  for inclusive quantities.

The impact of the input uncertainties in eq. (1) on the theoretical prediction of the pseudo-observables in Tab. 1 is given in Tab. 2. Here the numbers in brackets refer to the optimistic scenario for the theory error of  $\Delta\alpha$ . As evident from the table, the limiting factors among the input parameters are  $\alpha_s$  and  $\Delta\alpha$ . The present estimation of the parametric errors can in some cases exceed the anticipated experimental FCC-ee precision, but not by more than a factor of about 2.

Note that the quoted impact of  $\alpha_s$  on  $R_\ell$  is a somewhat circular statement, since  $R_\ell$  is the most important pseudo-observable for the determination of  $\alpha_s$ .

Quantity	FCC-ee	future parametric unc.	Main source
$M_W$ [MeV]	0.5 – 1	1 (0.6)	$\delta(\Delta\alpha)$
$\sin^2 \theta_{\text{eff}}^\ell$ [ $10^{-5}$ ]	0.6	2 (1)	$\delta(\Delta\alpha)$
$\Gamma_Z$ [MeV]	0.1	0.1 (0.06)	$\delta\alpha_s$
$R_b$ [ $10^{-5}$ ]	6	< 1	$\delta\alpha_s$
$R_\ell$ [ $10^{-3}$ ]	1	1.3 (0.7)	$\delta\alpha_s$

Table 2: Estimated experimental precision for the direct measurement of several important electroweak precision observables at FCC-ee [1, 2, 33] (column two, including systematic uncertainties). Third column: parametric uncertainty of several important EWPO due to uncertainties of input parameters given in (1), with the main source indicated in the fourth column.

As discussed above, as total uncertainty for the theoretical prediction of an observable the (quadratic) sum of parametric uncertainties plus intrinsic uncertainty should be taken<sup>6</sup>, as given in the fourth column of Tab. 1 and the second and third columns of Tab. 2. More generally, for combined fits to several observables, the parametric uncertainties should be taken into account separately by using the corresponding parameters in the fit.

The above numbers have all been obtained assuming the SM as calculational framework. The SM constitutes the model in which highest theoretical precision for the predictions of EWPO can be obtained. As soon as physics beyond the SM (BSM) will be discovered, an evaluation of the EWPO in any preferred BSM model will be necessary. The corresponding theory uncertainties, both intrinsic and parametric, can then be larger (see, e.g., [35, 49] for the Minimal Supersymmetric SM). A dedicated theory effort (beyond the SM) would be needed in this case.

### 4.3 Higgs precision observables

For the accurate study of the properties of the Higgs boson, precise predictions for the various partial decay widths, the branching ratios (BRs) and the Higgs-boson production cross sections along with their theoretical uncertainties are indispensable.

#### 4.3.1 Higgs-boson production cross-sections

The very narrow width of the Higgs boson allows for a factorization of all cross-sections with resonant Higgs bosons into production and decay parts to very high precision if the Higgs boson can be fully reconstructed. In this case, finite-width effects and off-shell contributions are of relative size  $\Gamma_H/M_H \sim 0.00003$  and thus not relevant; this is in contrast to physics with  $Z$  or  $W$  resonances, where  $\Gamma/M \sim 0.03$ . If the Higgs boson is not fully reconstructable

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<sup>6</sup>It should be noted that the intrinsic theory error is not a Gaussian random variable, which plays a role in the combination with other error sources.

Partial width	QCD	electroweak	total
$H \rightarrow b\bar{b}/c\bar{c}$	$\sim 0.2\%$	$< 0.3\%$	$< 0.4\%$
$H \rightarrow \tau^+\tau^-/\mu^+\mu^-$	–	$< 0.3\%$	$< 0.3\%$
$H \rightarrow gg$	$\sim 3\%$	$\sim 1\%$	$\sim 3.2\%$
$H \rightarrow \gamma\gamma$	$< 0.1\%$	$< 1\%$	$< 1\%$
$H \rightarrow Z\gamma$	$\lesssim 0.1\%$	$\sim 5\%$	$\sim 5\%$
$H \rightarrow WW/ZZ \rightarrow 4f$	$< 0.5\%$	$< 0.3\%$	$\sim 0.5\%$

Table 3: Current intrinsic uncertainties in the various Higgs-boson decay width calculations, see text and Refs. [19, 53].

(e.g. in  $H \rightarrow WW \rightarrow 2\ell 2\nu$ ) Higgs off-shell contributions have to be taken into account (which is straightforward at NLO).

At FCC-ee with  $\sqrt{s} = 240$  GeV (or other  $e^+e^-$  machines near this center-of-mass energy), the Higgs-boson production cross-section is strongly dominated by  $e^+e^- \rightarrow ZH$ , and  $e^+e^- \rightarrow \nu\bar{\nu}H$  contributes less than 20% [1, 5]. For these two processes full one-loop corrections in the SM are available [50, 51]. For the dominating  $ZH$  production mode they are found at the level of  $\sim 5$ –10%. It can be expected that missing two-loop corrections in the SM lead to an intrinsic uncertainty of  $\mathcal{O}(1\%)^7$ . This number has to be compared to the anticipated experimental accuracy of 0.4% [1, 2]. It becomes clear that with a full two-loop calculation of  $e^+e^- \rightarrow ZH$  the intrinsic uncertainty will be sufficiently small. Computational techniques for  $2 \rightarrow 2$  processes at the two-loop level exist, and it is reasonable to assume that, if required, this calculation within the SM can be incorporated for the FCC-ee Higgs precision studies.

For WBF production, the calculation of the full two-loop corrections will be significantly more difficult, since this is a  $2 \rightarrow 3$  process. However, one may assume that a partial result based on diagrams with closed light-fermion loops and top-quark loops (in a large- $m_t$  approximation) can be achieved, which should reduce the intrinsic theory uncertainty to below the 1% level. Given the fact that the WBF process is less crucial than the  $HZ$  channel for the Higgs physics program FCC-ee with  $\sqrt{s} = 240$  GeV, this will probably be adequate for most practical purposes.

### 4.3.2 Higgs-boson decays

The current intrinsic uncertainties for the various Higgs-boson decay widths are given in Tab. 3. They have been evaluated as follows [19]:

The QCD uncertainty for  $H \rightarrow q\bar{q}$  is assumed to be equal to the magnitude of the  $\mathcal{O}(\alpha_s^4)$  corrections [54]. The uncertainty due to missing  $\mathcal{O}(\alpha^2)$  contributions is estimated to be smaller than the known one-loop corrections [55], which themselves are unusually small due to accidental cancellations. Two-loop corrections of  $\mathcal{O}(\alpha\alpha_s)$  are also available [56] and the uncertainty from 3-loop mixed QCD-weak corrections is estimated to be of similar size as

<sup>7</sup>This estimate is corroborated by the recent calculation of the two-loop  $\mathcal{O}(\alpha\alpha_s)$  corrections to  $ZH$  cross-section [52], which were found to amount to 1.3%.

decay	para. $m_q$	para. $\alpha_s$	para. $M_H$
$H \rightarrow b\bar{b}$	1.4%	0.4%	–
$H \rightarrow c\bar{c}$	4.0%	0.4%	–
$H \rightarrow \tau^+\tau^-$	–	–	–
$H \rightarrow \mu^+\mu^-$	–	–	–
$H \rightarrow gg$	< 0.2%	3.7%	–
$H \rightarrow \gamma\gamma$	< 0.2%	–	–
$H \rightarrow Z\gamma$	–	–	2.1%
$H \rightarrow WW$	–	–	2.6%
$H \rightarrow ZZ$	–	–	3.0%

Table 4: Current parametric uncertainties in the various Higgs-boson decay width predictions [19] (see text). “–” indicates a negligible source of uncertainty.

the partial result in Ref. [57].

For  $H \rightarrow gg$ , the QCD uncertainty is estimated from the scale variation of the available N<sup>3</sup>LO corrections [58]. The electroweak uncertainty for this channel is estimated based on the observation that the NLO result [59] is dominated by light-fermion loops, and thus the NNLO contribution is expected to be suppressed by a factor  $N_{\text{lf}}\alpha \sim 0.1\text{--}0.2$ . The same procedure has been employed for  $H \rightarrow \gamma\gamma$ , using the results from Ref. [60]. Based on the experience from existing results for  $H \rightarrow gg$  and  $H \rightarrow \gamma\gamma$ , the currently unavailable electroweak NLO corrections to  $H \rightarrow Z\gamma$  are estimated to be less than 5%. Off-shell effects for  $H \rightarrow Z^*\gamma$  are known at the LO one-loop level [61] and the NLO corrections are expected to be small compared to the experimental uncertainty.<sup>8</sup>

The uncertainty due to the missing QCD and electroweak two-loop corrections for  $h \rightarrow WW, ZZ$  is estimated by (i) taking square of the known one-loop corrections [62] and, alternatively, (ii) doubling the numerical result of the known leading two-loop corrections in the large- $m_t$  limit [63].

Also the parametric uncertainties can play a non-negligible role for the evaluation of the partial widths. The most important parameters are the bottom quark mass and the strong coupling constant. In Ref. [53] the current uncertainties of  $\alpha_s$  and  $m_b$  have been assumed to be  $\delta\alpha_s = 0.0015$  and  $\delta m_b = 0.03$  GeV. Additionally, we consider  $\delta m_c = 0.025$  GeV,  $\delta m_t = 0.85$  GeV and  $\delta M_H = 0.24$  GeV [64]. The effect on the various partial widths has been evaluated as in Ref. [19] and is shown in Tab. 4.

When comparing the combined intrinsic and parametric uncertainties with the target precision of FCC-ee [1,2], see Tab. 5, it is clear that improvements are necessary. Concerning the intrinsic theory uncertainty, the available predictions for the  $f\bar{f}$  and  $\gamma\gamma$  channels are already sufficiently precise to match the expected FCC-ee experimental uncertainty. With available calculational techniques, the evaluation of complete two-loop corrections to  $H \rightarrow f\bar{f}$  can be achieved. This would reduce the uncertainty of the electroweak contributions to

<sup>8</sup> We assume that a proper experimental definition of this decay mode w.r.t. Dalitz decays [61] will be agreed upon.

decay	intrinsic	para. $m_q$	para. $\alpha_s$	para. $M_H$	FCC-ee prec. on $g_{HXX}^2$
$H \rightarrow b\bar{b}$	$\sim 0.2\%$	0.6%	$< 0.1\%$	–	$\sim 0.8\%$
$H \rightarrow c\bar{c}$	$\sim 0.2\%$	$\sim 1\%$	$< 0.1\%$	–	$\sim 1.4\%$
$H \rightarrow \tau^+\tau^-$	$< 0.1\%$	–	–	–	$\sim 1.1\%$
$H \rightarrow \mu^+\mu^-$	$< 0.1\%$	–	–	–	$\sim 12\%$
$H \rightarrow gg$	$\sim 1\%$		0.5% (0.3%)	–	$\sim 1.6\%$
$H \rightarrow \gamma\gamma$	$< 1\%$	–	–	–	$\sim 3.0\%$
$H \rightarrow Z\gamma$	$\sim 1\%$	–	–	$\sim 0.1\%$	
$H \rightarrow WW$	$\lesssim 0.3\%$	–	–	$\sim 0.1\%$	$\sim 0.4\%$
$H \rightarrow ZZ$	$\lesssim 0.3\%^\dagger$	–	–	$\sim 0.1\%$	$\sim 0.3\%$
$\Gamma_{\text{tot}}$	$\sim 0.3\%$	$\sim 0.4\%$	$< 0.1\%$	$< 0.1\%$	$\sim 1\%$

<sup>†</sup> From  $e^+e^- \rightarrow HZ$  production

Table 5: Projected intrinsic and parametric uncertainties for the partial and total Higgs-boson decay width predictions (see text). The last column shows the target of FCC-ee precisions on the respective coupling squared.

less than 0.1%. Similarly, the complete NLO corrections to  $H \rightarrow Z\gamma$  can be carried out with existing methods, resulting in an estimated precision of about 1% (see above for our estimate on the Dalitz decays).

More theoretical work is needed for  $H \rightarrow WW, ZZ, gg$ , which are currently limited by QCD uncertainties. For  $H \rightarrow WW, ZZ$ , the required QCD corrections are essentially identical to those for  $e^+e^- \rightarrow WW$ , and as explained on page 10 it is straightforward to improve them to a practically negligible level. Further significant progress would require the calculation of two-loop electroweak corrections, which for a  $1 \rightarrow 4$  process is beyond reach for the foreseeable future.

Note, however, that the  $HZZ$  coupling will be mostly constrained by the measurement of the  $e^+e^- \rightarrow HZ$  production process at FCC-ee with  $\sqrt{s} = 240$  GeV, rather than the decay  $H \rightarrow ZZ^*$ . As discussed in section 4.3.1, it may be assumed that full two-loop corrections (for on-shell  $Z$  and  $H$  bosons) will eventually be carried out for this process, leading to a remaining intrinsic uncertainty of less than 0.3%.

For  $H \rightarrow gg$ , the NNLO QCD corrections [65] and N<sup>3</sup>LO QCD corrections in the large- $m_t$  limit [58] are currently available. The leading uncertainty stems from the missing N<sup>4</sup>LO corrections in the large- $m_t$  limit. These require the calculation of massless four-loop QCD diagrams, which may be within reach [54, 66]. If these contributions become available, together with three-loop corrections involving bottom loops, the intrinsic uncertainty for  $H \rightarrow gg$  is expected to be reduced to the level of about 1%.

Also shown in Tab. 5 are the projected parametric uncertainties, assuming FCC-ee precisions, see Tab. 1. For inputs, we use  $\delta\alpha_s = 0.0002$  and  $\delta m_t = 50$  MeV from eq. (1),  $\delta M_H \sim 10$  MeV [67], and  $\delta m_b \sim 13$  MeV and  $\delta m_c \sim 7$  MeV [19].

The corresponding uncertainties (intrinsic, parametric from quark masses,  $\alpha_s$  and  $M_H$ ) for the total width are shown in the last line of Tab. 5. They are obtained by adding the

uncertainties for the partial widths linearly.

As discussed previously, as total uncertainty of an observable the sum of experimental, intrinsic and parametric uncertainty should be taken. Note that the numbers in Tab. 5 do not take into account correlations between the uncertainties in the Higgs production and decay processes or between different decay processes, in particular entering via  $\Gamma_{\text{tot}}$ . Their impact can only be evaluated when the full experimental correlation matrix is known. Most importantly, there is a strong correlation of the parametric uncertainty due to  $\delta m_b$  between  $g_{Hbb}^2$  and  $\Gamma_{\text{tot}}$ , and some partial correlation of the intrinsic uncertainties between  $g_{HVV}^2$  ( $V = W, Z$ ) and  $\Gamma_{\text{tot}}$ , see Refs. [53, 68, 69] for more details.

## 5 Conclusions

Due to the high anticipated experimental precision at the FCC-ee for electroweak and Higgs-boson precision measurements, theoretical uncertainties are expected to play an important role in the interpretation of these measurements within the Standard Model (SM), and thus for expected constraints on new physics. We have reviewed the status of current intrinsic and parametric uncertainties of EWPO and Higgs-boson precision observables, which are currently much larger than the anticipated future experimental uncertainties. In a second step we have evaluated the possible future intrinsic and parametric uncertainties.

The first one can be improved through calculations of higher-order corrections and the development of more advanced Monte-Carlo tools in the future. We aim to provide a guide of the possible impact and remaining uncertainty from future multi-loop calculations. To match the anticipated FCC-ee precision, one or two loop orders beyond what is available today will be required. It appears promising that some progress in this direction can be made by extending existing calculational techniques, but an extensive and concerted theory effort will be needed to achieve the ultimate FCC-ee precision goals. Similar considerations apply to other future  $e^+e^-$  colliders, such as ILC, CLIC, or CEPC.

The parametric uncertainties will improve due to the high-precision measurements of the SM input parameters obtained at the FCC-ee itself. Other proposed  $e^+e^-$  colliders have access to subset of these measurements with larger errors; the corresponding parametric uncertainties would have to be rescaled accordingly. These measurements also require that additional radiative corrections be calculated and implemented in Monte-Carlo tools to reduce the theory errors to the desired level.

We hope that this work helps to facilitate the corresponding efforts.

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