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ANALYSIS OF THE F. CALOGERO TYPE PROJECTION-ALGEBRAIC SCHEME FOR DIFFERENTIAL OPERATOR EQUATIONS

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Abstract

The existence, convergence, realizability and stability of solutions of differential operator equations obtained via a novel projection-algebraic scheme are analyzed in detail. This analysis is based upon classical discrete approximation techniques coupled with a recent generalization of the Leray-Schauder fixed point theorem. An example is included to illustrate the efficacy of the projection scheme and analysis strategy.

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1. Introduction

About twenty-six years ago, Calogero [4] developed a novel scheme for approximating the eigenvalues of differential operators - an approach that was shown to be very effective in several applications (see e.g. [3, 5, 6]). Shortly thereafter, Mitropolski et al. [17] devised a mathematically rigorous projection-algebraic framework for Calogero's method, which also expanded its applicability to a wider range of problems involving differentiable operators. The combined approach of these researchers, which we shall refer to as the *CMPS scheme*, appears capable of very effectively and efficiently generating approximate solutions to many dynamical equations of mathematical physics. Since the the pioneering work on the CMPS method, there have been several investigations of the range of applicability and convergence properties of this scheme, such as Calogero & Franco [6], Samoylenko [22], Calogero [5], Luśtyk [13, 14], Luśtyk & Bihun [15] and Bihun & Luśtyk [3]; all of which have provided glimpses of the effectiveness of the CMPS method for specific applications. However, several fundamental numerical analytic issues have not as yet been adequately resolved in the literature for extensive classes of differential operator equations, which means there are many outstanding questions concerning the efficacy of the CMPS scheme.

We take a major step here in addressing the unanswered questions by investigating the existence, convergence, realizability and stability of the CMPS method for differential operator equations of the form

$$Au = f(u), \tag{1.1}$$

where $f : X \to Y$ is a (nonlinear) continuous mapping between Banach (function) spaces X and Y. Here $A : X \to Y$ is a closed linear differential map, representable as

$$A := \sum_{|\beta|=0}^{m} a_\beta(x) \frac{\partial^\beta}{\partial x^\beta} \tag{1.2}$$

in an open region $\Omega \subset \mathbb{R}^q$ with smooth coefficients $a_\beta \in C^\infty(\Omega; \mathbb{R})$, where q, m belong to the set $\mathbb{N}$ of natural numbers (positive integers), defined on $\operatorname{dom} A \subset X$ and satisfying the condition $\overline{\operatorname{im} A} = Y$, which is trivially the case when $\operatorname{im} A = Y$. The notation used here is of the standard multi-index form, with $\beta := (\beta_1, \ldots, \beta_q) \in \mathbb{Z}_+^q$, where $\mathbb{Z}_+$ is the set of nonnegative integers, and $|\beta| := \beta_1 + \cdots + \beta_q$.

Projection based methods for constructing approximate solutions to systems of equations of various types have proven to be remarkably effective, efficient and robust for many applications [4, 7, 9, 11, 17, 18, 21, 24, 26]. A special class of these methods, usually referred to as the projection-algebraic approach, has been the focus of considerable research attention in the numerical analysis community over the last three decades, largely as a result of its inherent suitability to solving problems related to differential operator equations. An important contribution in this line of research was the 1983 paper of Calogero [4] describing a new method for calculating eigenvalues of linear differential operators on Hilbert spaces, which was subsequently (in 1988) given a projection-algebraic framework [17] that appears to be especially promising for numerous applications to dynamical systems arising in mathematical physics. Among the investigations that should also be noted in the development and applications of projection-algebraic schemes is the earlier work of Casas [7] and the more recent contribution of Wei & Norman [26]. Several studies of the efficiency and accuracy of the CMPS scheme have underscored and provided further evidence of the considerable potential of this approach. Among these are the investigations of Samoylenko [22] and Lustyk [13] indicating that the CMPS converges very rapidly in several applications, and a number of papers confirming the excellent approximation characteristics for a variety of classical initial/boundary-value problems, including those involving the nonhomogeneous heat equation [3, 13, 14, 15]. However, as mentioned above there are many fundamental questions concerning the CMPS approach that have not been satisfactorily resolved for a wide range of differential operator problems. Our intention here is to both further develop and refine the CMPS, and to answer most of these questions.

Our investigation in this paper is organized as follows. In Section 2, we formulate realizability, solvability and convergence criteria for the CMPS approach, and establish necessary conditions for these properties to obtain for this projection-algebraic scheme in a rather general context. Our methods of proof of these conditions build upon the work in [3, 14, 15, 17], and rely critically on a recent extension by Prykarpatsky [19, 20] of the Leray-Schauder fixed point theorem. Next, in Section 3, we employ a suitable closure of the universal enveloping algebra of the Heisenberg-Weil algebra (which comprises the Lie algebraic core of the CMPS method) for the constructive functional-interpolation component of the scheme (in terms of finite-dimensional quasi-representations), and show how this specificity leads to an enhanced convergence analysis for the scheme when the function f is assumed to be constant. We also show that in the special case when Lagrangian projectors and Lagrangian interpolation are applicable, we obtain even better convergence characteristics. The efficacy of the CMPS scheme is demonstrated by applying it to a substantial differential operator problem in Section 4. We conclude in Section 5 with some brief observations about our results and indications of related future research directions.

2. Functional-operator aspects of the CMPS scheme in Banach spaces

Consider the nonlinear operator equation (1.1), where $A : X \to Y$ is a closed surjective linear differential operator defined on $\mathrm{dom} A \subset X$ (not necessarily dense), and $f : X \to Y$ is an arbitrary nonlinear continuous mapping from the Banach space X to the Banach space Y with domain $\mathrm{dom}\, f = D_r(0) \cap \mathrm{dom} A \subset X$ (here $D_r(0) \subset X$ is the closed disk of radius $r > 0$ centered at zero). Let us assume that this equation satisfies the following properties:

i) the mapping is A-compact, that is for any bounded sets $U \subset \mathrm{dom} f$ and $V \subset Y$ the closure $\overline{f(U \cap A^{-1}(V))} \subset Y$ is compact;

ii) the $\dim \ker A \geq 1$;

iii) there exist positive numbers $k_f < k_A \in \mathbb{R}_+$, defined as and satisfying

$$k_A^{-1} := \sup_{\|v\|_Y = 1} \inf_{u \in \mathrm{dom} A} \{\|u\|_X : Au = v\} < \infty, \quad k_f := \sup_{u \in D_r(0)} \frac{1}{r} \|f(u)\| < \infty. \tag{2.1}$$

If $\tilde{X}_N \subset \tilde{X}_{N+1} \subset X$ and $\tilde{Y}_N \subset \tilde{Y}_{N+1} \subset Y$, $N \in \mathbb{Z}_+$, are suitable finite-dimensional Banach subspaces, and $P_N^{(x)} : X \to \tilde{X}_N$, $N \in \mathbb{Z}_+$, and $P_N^{(y)} : Y \to \tilde{Y}_N, N \in \mathbb{Z}_+$, are the corresponding projectors, one considers the following sequence of equations

$$P_N^{(y)} A \tilde{u}_N = P_N^{(y)} f(\tilde{u}_N) \tag{2.2}$$

on elements $\tilde{u}_N \in \tilde{X}_N$, $N \in \mathbb{Z}_+$, which are suitable approximations to a solution of equation (1.1) that is being sought. These solutions are, in general, non-unique, as $\dim \ker A \geq 1$.The projection method is often called *realizable* if the set $\mathcal{M} \subset X$ of solutions to equation (1.1) is nonempty, and for sufficiently large $N \in \mathbb{Z}_+$ there are nonempty sets $\mathcal{M}_N \subset \tilde{X}_N$ of solutions to equations (2.2). The method is called *convergent* if it is realizable and satisfies the property

$$\lim_{N \to \infty} \sup_{\tilde{u}_N \in \tilde{\mathcal{M}}_N} \inf_{u \in \mathcal{M}} ||\tilde{u}_N - u||_X = 0. \tag{2.3}$$

Obviously, the realizability criteria of the projection method and its convergence are extremely important when it comes to applications, so we shall first address these properties. The solvability aspect shall be disposed of using the recent generalization of the Leray-Schauder fixed point theorem for mappings between Banach spaces in [19, 20], where the following result was proved concerning the solution set $\mathcal{M}$ of the nonlinear operator equation (1.1).

Theorem 2.1. *Let conditions i) and ii), formulated above, hold. Then, owing to the condition* $\dim \ker A \geq 1$, *equation (1.1) possesses a nonempty solution set* $\mathcal{M}$, *whose topological dimension satisfies* $\dim \mathcal{M} \geq \dim \ker A - 1$.

In the sequel, we shall assume that all of the hypotheses of Theorem 2.1 are fulfilled. Then the following result characterizes the realizability of the projection method (2.2).

Theorem 2.2. *Let conditions i), ii) be fulfilled and additionally*

$$\lim_{N\to\infty} \sup_{v\in \operatorname{im} A\cap \operatorname{im} f} ||P_N^{(y)} v - v||_Y = 0. \tag{2.4}$$

Then for all sufficiently integers $N \in \mathbb{Z}_+$ the solution sets $\tilde{\mathcal{M}}_N \subset \tilde{X}_N$ are nonempty and the convergence condition (2.3) holds.

Proof. Define

$$k_f^{(N)} := \sup_{\tilde{u}_N\in D_r(0)} \frac{1}{r} ||P_N^{(y)} f(\tilde{u}_N)||_{\tilde{Y}_N}, \tag{2.5}$$

and

$$k_A^{(N),-1} := \sup_{\tilde{v}_N\in\tilde{Y}_N} \frac{1}{||\tilde{v}_N||_{\tilde{Y}_N}} \inf_{\tilde{u}_N\in P_N^{(x)}(\operatorname{dom} A)} \{||\tilde{u}_N||_{\tilde{X}_N} : P_N^{(y)} A\tilde{u}_N = \tilde{v}_N\}. \tag{2.6}$$

We can then choose an integer $N_0 \in \mathbb{Z}_+$ such that $\dim\ker(P_{N_0}^{(y)} A) \geq 1$, and

$$k_f \leq k_f^{(N_0)} < k_A^{(N_0)} \leq k_A. \tag{2.7}$$

Then based on expressions (2.5) and (2.6) from condition (2.7), we obtain the following inequalities for all $N \geq N_0$:

$$k_f \leq k_f^{(N)} < k_A^{(N)} \leq k_A \tag{2.8}$$

But this means that, owing to the generalized Leray-Schauder type fixed point theorem [19, 20], the sequence of equations (2.3) possesses solutions for all $N \geq N_0$; that is, all solution sets $\tilde{\mathcal{M}}_N \subset \tilde{X}_N$, $N \geq N_0$ are nonempty, and the projection method itself is realizable.

Now choose $\varepsilon > 0$ and consider the neighborhood

$$U_\varepsilon(\mathcal{M}) := \{u \in \operatorname{dom} f : \inf_{\bar{u}\in\mathcal{M}\subset\operatorname{dom} f} ||\bar{u} - u||_X < \varepsilon. \tag{2.9}$$

It is evident that the closed set $\operatorname{dom} f \setminus U_\varepsilon(\mathcal{M})$ does not contain solutions to equation (1.1), and for some $\alpha_\varepsilon > 0$ we have

$$\inf_{\bar{u}\in\operatorname{dom} f\setminus U_\varepsilon(\mathcal{M})} ||Au - f(u)||_Y = \alpha_\varepsilon > 0. \tag{2.10}$$

Choose now, based on (2.4), an integer $N_\varepsilon \in \mathbb{Z}_+$ such that

$$\sup_{u\in\operatorname{dom} f} (||Au - P_N^{(y)} Au||_Y + ||f(u) - P_N^{(y)} f(u)||_Y) < \alpha_\varepsilon \tag{2.11}$$

for all $N \geq N_\varepsilon$. Then for all $u \in \operatorname{dom} f \setminus U_\varepsilon(\mathcal{M})$, the following inequality

$$\begin{gathered} ||P_N^{(y)} Au - P_N^{(y)} f(u)||_Y \geq ||Au - f(u)||_Y - \\ -(||Au - P_N^{(y)} Au||_Y + ||f(u) - P_N^{(y)} f(u)||_Y) > \alpha_\varepsilon - \alpha_\varepsilon = 0, \end{gathered}$$

holds. Hence, we have the embeddings $\tilde{\mathcal{M}}_N \subset U_\varepsilon(\mathcal{M})$ for all $N \geq N_\varepsilon$. Finally, by choosing $\varepsilon > 0$ small enough, we can insure that the condition $\tilde{\mathcal{M}}_N \subset U_\varepsilon(\mathcal{M})$ for all $N \geq N_\varepsilon$ is equivalent to that of convergence for (2.3), thus completing the proof. □

In the case when the sequences of subspaces $\tilde{X}_N \subset \tilde{X}_{N+1} \subset X, N \in \mathbb{Z}_+$ and $\tilde{Y}_N \subset \tilde{Y}_{N+1} \subset Y, N \in \mathbb{Z}_+$, are Hilbert spaces and, moreover

$$\cup_{N\in\mathbb{Z}_+} \tilde{X}_N = X, \qquad \cup_{N\in\mathbb{Z}_+} \tilde{Y}_N = Y, \tag{2.12}$$

with orthogonal projectors $P_N^{(x)} : X \to \tilde{X}_N, P_N^{(y)} : Y \to \tilde{Y}_N, N \in \mathbb{Z}_+$, the norms $||P_N^{(x)}|| = 1$, $||P_N^{(x)}|| = 1, N \in \mathbb{Z}_+$,and for all $u \in X, v \in Y$ we have

$$(2.13)\qquad \lim_{N\to\infty} ||u - P_N^{(x)} u||_Y = 0, \qquad \lim_{N\to\infty} ||v - P_N^{(y)} v||_Y = 0.$$

If we assume further that conditions (2.3), (2.4) are fulfilled and $\dim\ker A \geq 1$, then we obtain the following analog of Theorem 2.2 for realizability of the projection-algebraic scheme for the nonlinear operator equation (1.1) in Hilbert spaces.

Theorem 2.3. *For all sufficiently large $N \in \mathbb{Z}_+$, the solution sets $\tilde{\mathcal{M}}_N \subset \tilde{X}_N$ are nonempty and the convergence condition (2.2) holds.*

Proof. It is clear that we need only verify the condition (2.4). Having assumed the contrary, one can find a subsequence of indices $N_k \in \mathbb{Z}_+$ for $k \in \mathbb{Z}_+$, as well as elements $u_k \in \operatorname{dom} f, k \in \mathbb{Z}_+$, for which there exists $\varepsilon > 0$ such that

$$(2.14)\qquad ||P_{N_k}^{(y)} f(u_k) - f(u_k)||_Y > \varepsilon.$$

Since for all $k \in \mathbb{Z}_+$ the elements $f(u_k) \in \operatorname{im} A$, owing to the A-compactness of the mapping $f : \operatorname{dom} f \to Y$, we must have $\lim_{k\to\infty} f(u_k) = \bar{v} \in Y$. Making use of the limits (2.13), we obtain:

$$\lim_{k\to\infty} ||P_{N_k}^{(y)} f(u_k) - f(u_k)||_Y \leq \lim_{k\to\infty} ||P_{N_k}^{(y)} f(u_k) - P_{N_k}^{(y)} \bar{v}||_Y +$$
$$+ \lim_{k\to\infty} ||P_{N_k}^{(y)} \bar{v} - \bar{v}||_Y + \lim_{k\to\infty} ||\bar{v} - f(u_k)||_Y = 0,$$

contradicting the initial inequality (2.14), and proving the theorem. □

If the mapping $f : \operatorname{dom} f \subset X \to Y$ is constant, the operator $A : \operatorname{dom} A \subset X \to Y$ is densely defined and $\overline{\operatorname{im} A} = Y$, and one can obtain additional convergence conditions for the CMPS scheme for equation (1.1), which we investigate in what follows.

3. Functional-interpolation properties of the CMPS method

3.1. **Lie-algebraic preliminaries.** Here we present some preliminaries from the theory of Lie algebraic structures [23] and the Lagrangian interpolation concepts [1] necessary for our further exposition of the scheme. We assume here, for brevity, that the mapping $f : X \to Y$ is a constant function, and note that the more general case can be disposed of in an entirely analogous way requiring only the essentially obvious additional details. It also shall be assumed that the differential operator $A(x;\partial) : X \to Y$ belongs to a suitable operator closure of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of the Heisenberg-Weil algebra $\mathfrak{g} = \oplus_{1\leq j\leq q}\mathfrak{g}_j$, $\mathfrak{g}_j := \{x_j, \partial_{x_j}, 1\}$, $j = 1, ..., q$, of differential operations [10, 23]. Here $x_j : X \to X$ is the operator of multiplication on the independent variable $x_j \in \mathbb{R}$, ∂_{x_j} represents partial differentiation with respect to variable the $x_j, 1 \leq j \leq q$, and $1 : X \to X$ is the identity operator. The Lie bracket of the Lie algebra $\mathfrak{g}$ is defined to be the standard commutator: $[a, b] := a \cdot b - b \cdot a$ for any elements $a, b \in \mathcal{U}(\mathfrak{g})$, where " $\cdot$ " denotes the usual superposition of operators.

As mentioned above, when using the projection-algebraic method (CMPS), we try to find the corresponding representations of all elements involved in (1.2) - both functions and operators [17]. We define a sequence of linear mappings $\Phi_N^{(x)} := \pi_N^{(x)} P_N^{(x)} : X \to X_N$, $\Phi_N^{(y)} := \pi_N^{(y)} P_N^{(y)} :\to Y_N$, $N \in \mathbb{Z}_+^q$, where $P_N^{(x)} : X \to \tilde{X}_N \subset X$ and $P_N^{(y)} : Y \to \tilde{Y}_N \subset Y$ are suitable projection operators defined on finite-dimensional functional subspaces $\tilde{X}_N \subset X$, $N \in \mathbb{Z}_+^q$, and $\tilde{Y}_N \subset Y$, $N \in \mathbb{Z}_+^q$, respectively, of vector-function polynomials in the variables $x_j \in \mathbb{R}$, $1 \leq j \leq q$, and $\pi_N^{(x)} : \tilde{X}_N \rightleftharpoons X_N$, $\pi_N^{(y)} : \tilde{Y}_N \rightleftharpoons X_N$ are the corresponding isomorphisms between functional subspaces $\tilde{X}_N \subset X$, $\tilde{Y}_N \subset Y$, and corresponding finite-dimensional Euclidean spaces X_N, Y_N, respectively, satisfying the following conditions: $\dim \tilde{X}_N = \dim X_N = \dim Y_N = \dim \tilde{Y}_N$ for all $N \in \mathbb{Z}_+^q$. In order to define the projectors $P_N^{(x)} : X \to \tilde{X}_N \subset X$ and $P_N^{(y)} : Y \to \tilde{Y}_N \subset Y$ more precisely, we consider a lattice Θ of an open cube $\Omega := K \subset \mathbb{R}^q$ with nodes that are mesh points with respect to variables $x \in K \subset \mathbb{R}^q$, that is

$$(3.1)\qquad \Theta := \{x_{(i)} \in \Omega : (i) \in \mathbb{Z}_+^q\}.$$

Then, by definition,

$$P_N^{(x)}u := \sum_{(i)} L_{(i)}(x)u(x_{(i)}), \quad P_N^{(y)}v := \sum_{(i)} L_{(i)}(x)v(x_{(i)}), \tag{3.2}$$

for arbitrary continuous functions $u \in \tilde{X}_N \subset X$ and $v \in \tilde{Y}_N \subset Y$, where the $L_{(i)}(x) := \otimes_{1\leq j\leq q} l_j(x_j|x_{(i)})$, $(i) \in \mathbb{Z}_+^q$,are basis Lagrange polynomials, normalized by the unity operator:

$$\sum_{(i)} L_{(i)}(x) := I := \otimes_{1\leq j\leq q} 1_j, \tag{3.3}$$

where $\otimes$ is the usual tensor product of vectors.

Let $S_N^{(j)}, D_N^{(j)}$and $I_N^{(j)} \in Hom(X_N; Y_N)$, $j = 1,\dots,q$, $N \in \mathbb{Z}_+^q$, be the corresponding matrix quasi-representations [17] of the Heisenberg-Weil algebra basis with respect to the Lagrange interpolation mappings (3.2). Then problem (1.1) can be represented as the following sequence of linear algebraic vector equations:

$$A_N u_N = f_N, \ , \tag{3.4}$$

where $A_N := A(S_N; D_N)$ is defined as a special selection [16] of the set-valued mapping $\Phi_N^{(y)} A \Phi_N^{(x),-1} : X_N \to Y_N$, $u_N := \Phi_N^{(x)} u \in X_N$, $f_N := \Phi_N^{(y)} f \in Y_N$, $N \in \mathbb{Z}_+^q$.

The sequence of algebraic vector equations (3.4) is the key ingredient for studying the construction of approximate solutions to the differential-operator equation (1.1) using the CMPS.

3.2. **Convergence analysis.** We now consider two families of finite-dimensional functional subspaces $\tilde{X}_N \subset X$ and $\tilde{Y}_N \subset Y$ for $N \in \mathbb{Z}_+$, that are chosen as described above:

$$\begin{aligned} \tilde{X}_N &\subset \tilde{X}_{N+1}, & \tilde{Y}_N &\subset \tilde{Y}_{N+1}, \\ \overline{\cup_{N\in\mathbb{Z}_+} \tilde{X}_N} &= X, & \overline{\cup_{N\in\mathbb{Z}_+} \tilde{Y}_N} &= Y \end{aligned} \tag{3.5}$$

Assume, for brevity, that the region $\Omega \subset \mathbb{R}^q$ is bounded. Then for the space $X := L_p(\Omega; R)$ and domain $\operatorname{dom} A = W_p^{(m+s)}(\Omega)$ and $\operatorname{im} A = W_p^{(s)}(\Omega) \subset L_p(\Omega; R) := Y$, $p > q$, $s > 0$, we have expressions

$$\begin{aligned} \tilde{X}_N &:= P_N^{(x)} W_p^{(m+s)}(\Omega), \\ \tilde{Y}_N &:= P_N^{(y)} W_p^{(s)}(\Omega), \end{aligned} \tag{3.6}$$

where the $P_N^{(x)}$ are linear operators defined on the space of continuous functions on $\Omega \subset \mathbb{R}^q$. It is well known that the operators $P_N^{(x)}$ and $P_N^{(y)}$ are projectors satisfying the conditions

$$P_N^{(x)} P_N^{(x)} = P_N^{(x)}, \quad P_N^{(y)} P_N^{(y)} = P_N^{(y)} \tag{3.7}$$

for all $N \in \mathbb{Z}_+$.

Consider now for each $N \in \mathbb{Z}_+$ the following equation

$$P_N^{(y)} A \tilde{u}_N = P_N^{(y)} f \tag{3.8}$$

on an element $\tilde{u}_N \in \tilde{X}_N$, for which as $N \to \infty$

$$\lim_{N\to\infty} \|A\tilde{u}_N - f\|_Y = 0, \tag{3.9}$$

where mapping $f : X \to Y$ is a constant element of the space Y. It is evident that equation (3.8) possesses a unique solution $\tilde{u}_N \in \tilde{X}_N$, if the following equality holds for each $N \in \mathbb{Z}_+$:

$$P_N^{(y)} A \tilde{X}_N = \tilde{Y}_N. \tag{3.10}$$

But (3.10) is equivalent to the existence of the inverse finite-dimensional operator

$$P_N^{(y)} A P_N^{(x)} := A_N : \tilde{X}_N \to \tilde{Y}_N \tag{3.11}$$

for every $N \in \mathbb{Z}_+$.

The concept of an arbitrary limiting-dense family of subspaces of a Banach space shall prove useful in the sequel.

Definition 3.1. A family of subspaces $\{\mathcal{B}_N \subset \mathcal{B} : N \in \mathbb{Z}_+\}$ is called *limiting-dense* in B, if for each $g \in B$,

$$\rho(g, \mathcal{B}_N) := \inf_{\tilde{w}_N \in \mathcal{B}_N} \|g - \tilde{w}_N\|_{\mathcal{B}} \to 0 \tag{3.12}$$

as $N \to \infty$.

In order to continue with our analysis, we shall need the following convergence theorem for our approximation process, which is a slight generalization of the corresponding result in [12].

Theorem 3.2. *Let the linear operator $A : X \to Y$ with a dense domain $\operatorname{dom} A \subset X$ be invertible and satisfy the condition $\overline{\operatorname{im} A} = Y$, where X and Y are Banach spaces. Assume also that a family of subspaces $\left\{A\tilde{X}_N \in Y : \ N \in \mathbb{Z}_+\right\}$ is limiting-dense and projection operators $P_N^{(y)} : Y \to \tilde{Y}_N \subset Y$ satisfy the condition*

$$\|P_N^{(y)}\| \leq c_N^{(y)} \tag{3.13}$$

for some positive sequence $c_N^{(y)} \in \mathbb{R}_+$, $N \in \mathbb{Z}_+$. Then for each element $f \in Y$, equations

$$P_N^{(y)} A u = P_N^{(y)} f \tag{3.14}$$

have the unique solutions $\tilde{u}_N \in \tilde{X}_N$ for all $N \in \mathbb{Z}_+$, where

$$\lim_{N \to \infty} \|A\tilde{u}_N - f\|_Y = 0, \tag{3.15}$$

iff
i) condition (3.10) is satisfied;
ii) there exists a positive sequence $\tau_N^{(y)} \in \mathbb{R}_+$, $N \in \mathbb{Z}_+$, such that

$$\|P_N^{(y)} \tilde{v}_N\|_{\tilde{Y}_N} \geq \tau_N^{(y)} \|\tilde{v}_N\|_Y, \quad \overline{\lim_{N \to \infty}} (c_N^{(y)}/\tau_N^{(y)}) < \infty \tag{3.16}$$

for each element $\tilde{v}_N \in A\tilde{X}_N$, $N \in \mathbb{Z}_+$;
iii) the limit supremum satisfies

$$\overline{\lim_{N \to \infty}} \left[\left(1 + (c_N^{(y)}/\tau_N^{(y)})\right) \rho(f, A\tilde{X}_N)\right] = 0 \tag{3.17}$$

for every $f \in Y$.

Proof. First, assume that for each element $f \in Y$ equation $P_N^{(y)} Au = P_N^{(y)} f$, $N \in \mathbb{Z}_+$, has the unique solution $\tilde{u}_N \in \tilde{X}_N$, and that $\|A\tilde{u}_N - f\|_Y \to 0$ as $N \to \infty$. Then, it follows from

$$\rho(f, A\tilde{X}_N) = \inf_{w_N \in A\tilde{X}_N} \|f - \tilde{w}_N\|_Y \leq \|f - A\tilde{u}_N\|_Y$$

that $\lim_{N\to\infty} \rho(f, A\tilde{X}_N) = 0$, so the family of subsets $\left\{A\tilde{X}_N \in Y : N \in \mathbb{Z}_+\right\}$ is limiting-dense in Y. Choose $N \in \mathbb{Z}_+$ and consider $P_N^{(y)} Au = \tilde{f}_N \in \tilde{Y}_N$. It is clear that there exists an element $f \in Y$ for which $P_N^{(y)} f = \tilde{f}_N$, ensuring, owing to the hypotheses, the existence of a unique solution $\tilde{u}_N \in \tilde{X}_N$. But this is tantamount to $P_N^{(y)} A\tilde{X}_N = \tilde{Y}_N$, which proves condition i).

Since the mapping $P_N^{(y)} : Y \to \tilde{Y}_N \subset Y$ is a projector, one can consider its restriction $\bar{P}_N^{(y)} := P_N^{(y)}|_{A\tilde{X}_N} : A\tilde{X}_N \to \tilde{Y}_N$ for each $N \in Z_+$. It follows from (3.13) that the operator $\bar{P}_N^{(y)} : A\tilde{X}_N \subset Y \to \tilde{Y}_N$, is a bounded injective mapping. Consequently, the Banach inverse operator theorem [2, 10, 25] implies that there exists the bounded inverse operator $\bar{P}_N^{(y),-1}$: $\tilde{Y}_N \to A\tilde{X}_N \subset Y$.

Let now $\tilde{u}_N \in \tilde{X}_N$ be the corresponding approximate solution of the equation $P_N A u = P_N f$. Then, $A\tilde{u}_N = \bar{P}_N^{(y),-1} P_N f$, whence from condition (3.15) one concludes that

$$\lim_{N\to\infty} \|\bar{P}_N^{(y),-1} P_N f - f\|_Y = 0 \tag{3.18}$$

for any $f \in Y$, which means that $\lim_{N\to\infty} \bar{P}_N^{(y),-1} P_N f = f$ for every given element $f \in Y$. Hence, making use of the classical Banach-Steinhaus theorem [1, 2, 10, 25] we obtain that

$$\sup_{N\in\mathbb{Z}_+} \|\bar{P}_N^{(y),-1} P_N^{(y)}\|_Y \leq c^{(y)} < \infty \tag{3.19}$$

for some $c^{(y)} \in \mathbb{R}_+$. Thus, for each element $\tilde{w}_N = P_N^{(y)} j_N \tilde{w}_N \in \tilde{Y}_N$, where $j_N : \tilde{Y}_N \to Y$ is the corresponding imbedding operator, one finds that

$$\|\bar{P}_N^{(y),-1} \tilde{w}_N\|_Y = \|\bar{P}_N^{(y),-1} P_N^{(y)} j_N \tilde{w}_N\|_Y \leq \|\bar{P}_N^{(y),-1} P_N\|_Y \|j_N \tilde{w}_N\|_Y \leq c^{(y)} \|j_N\| \|\tilde{w}_N\|_Y \tag{3.20}$$

for all $N \in \mathbb{Z}_+$. But this means that the norm of the operator $\bar{P}_N^{(y),-1} : \tilde{Y}_N \to A\tilde{X}_N \subset Y$ is bounded for all $N \in \mathbb{Z}_+$, that is

$$||\bar{P}_N^{(y),-1}|| \leq c^{(y)} ||j_N||. \tag{3.21}$$

Choose now an arbitrary element $\tilde{v}_N \in A\tilde{X}_N \subset Y$ and calculate $\tilde{w}_N := \bar{P}_N^{(y)} \tilde{v}_N \in \tilde{Y}_N$. Then, making use of the inequality (3.21), we obtain

$$\|\tilde{v}_N\|_Y = \|\bar{P}_N^{(y),-1} \tilde{w}_N\|_Y \leq c^{(y)} \|j_N\| \|\tilde{w}_N\|_Y := \tau_N^{(y),-1} \|P_N \tilde{v}_N\|_{\tilde{Y}_N}, \tag{3.22}$$

where the quantities $\tau_N^{(y),-1} := c^{(y)} \|j_N\| > 0$ are bounded for all $N \in \mathbb{Z}_+$.This means that the condition $ii)$ is fulfilled for each element $\tilde{v}_N \in A\tilde{X}_N$, that is $\|P_N^{(y)} \tilde{v}_N\| \geq \tau_N^{(y)} \|\tilde{v}_N\|_Y$, $N \in \mathbb{Z}_+$.

To prove the sufficiency of conditions $i) - iii)$ we shall argue as follows: Let us solve the equation $P_N A u = P_N f$ for $N \in \mathbb{Z}_+$, whose solution $\tilde{u}_N \in \tilde{X}_N$ is unique, and can be represented as

$$\tilde{u}_N = A^{-1} \bar{P}_N^{(y),-1} P_N f, \tag{3.23}$$

where, as above, the linear mapping $\bar{P}_N^{(y)} := P_N^{(y)}|_{A\tilde{X}_N} : A\tilde{X}_N \to \tilde{Y}_N$ is the corresponding restriction upon $A\tilde{X}_N \subset Y$ of the projection operator $P_N^{(y)} : Y \to Y$ on the subspace $\tilde{Y}_N \subset Y$. Owing to condition $ii)$, we have $\|\bar{P}_N^{(y)-1}\|_Y \leq \tau_N^{(y),-1}$, and the norm $\|\bar{P}_N^{(y),-1} P_N^{(y)}\| \leq c_N^{(y)} \tau_N^{(y),-1}$ for all $N \in \mathbb{Z}_+$. Whence, for any element $\tilde{w}_N \in A\tilde{X}_N \subset Y$ we compute that

$$\begin{aligned} \|A\tilde{u}_N - f\|_Y = \|\bar{P}_N^{(y),-1} P_N f - f\|_Y \leq \\ \leq \inf_{\tilde{w}_N \in A\tilde{X}_N} \left(\|\bar{P}_N^{(y),-1} P_N f - \bar{P}_N^{(y),-1} P_N^{(y)} \tilde{w}_N\|_Y + \|\tilde{w}_N - f\|_Y \right) \leq \\ \leq \inf_{\tilde{w}_N \in A\tilde{X}_N} \left(\|\bar{P}_N^{(y),-1} P_N f - \bar{P}_N^{(y),-1} P_N^{(y)} \tilde{w}_N\|_Y + \|\tilde{w}_N - f\|_Y \right) \leq \\ \leq \inf_{\tilde{w}_N \in A\tilde{X}_N} \left(c_N^{(y)} \tau_N^{(y),-1} + 1 \right) \rho(f, \tilde{w}_N) = \left(c^{(y)} \tau_N^{(y),-1} + 1 \right) \rho(f, A\tilde{X}_N), \end{aligned} \tag{3.24}$$

where we took into account that $\bar{P}_N^{(y),-1} P_N^{(y)} \tilde{w}_N = \tilde{w}_N$ for all $\tilde{w}_N \in A\tilde{X}_N \subset Y$. Then assumption $iii)$ implies the existence of the limit $\lim_{N\to\infty} \|A\tilde{u}_N - f\|_Y = 0$ for an arbitrary element $f \in Y$, so the proof is complete. □

Remark. We note here that an (alternative) analog of Theorem 3.2, was stated in [12].

There is an obvious corollary that follows directly from the proof of Theorem 3.2 in the case when $\dim \tilde{X}_N = \dim \tilde{Y}_N < \infty$ for all $N \in \mathbb{Z}_+$, to wit, we see that condition $i)$ in form (3.10) follows from $ii)$. Moreover, we have the next result about the convergence of the solutions $\tilde{u}_N \in \tilde{X}_N$ to an element $u \in X$ as $N \to \infty$.

Theorem 3.3. *Let the hypotheses of Theorem 3.2 be fulfilled; in particular, the closed operator $A : X \to Y$ is surjective (so that $\|A^{-1}\| < \infty$ owing to the classical results* [2, 10, 25] *concerning closed operators). Then the sequence of solutions $\tilde{u}_N \in \tilde{X}_N$ of the equations $P_N^{(y)} Au = P_N^{(y)} f$ as $N \to \infty$ are approximations - that converge - in the norm $\| \cdot \|_X$ to a solution of the equation $Au = f$.*

Proof. Assume that $u_N \in X_N$ is a solution to the equation $P_N^{(y)} Au_N = P_N^{(y)} f$ for all $N \in Z_+$. Then one can estimate the difference $(u - \tilde{u}_N\ ) \in X$ in the norm in the Banach space X as follows:

$$
\begin{aligned}
\|\tilde{u}_N - u\|_X = \|\tilde{u}_N - A^{-1}f\|_X = \|A^{-1}A\tilde{u}_N - A^{-1}f\|_X = \\
= \|A^{-1}(A\tilde{u}_N - f)\|_X \leq \|A^{-1}\| \ \|A\tilde{u}_N - f\|_Y.
\end{aligned}
\tag{3.25}
$$

Then, in virtue of the inequality (3.24), we conclude that $\lim_{N\to\infty} \|A\tilde{u}_N - f\|_Y = 0$. As the inverse operator A^{-1} is closed and, therefore bounded, the right-hand side of inequality (3.25) tends to zero as $N \to \infty$. Consequently, $\lim_{N\to\infty} \|\tilde{u}_N - u\|_X = 0$, and the proof is complete. □

3.3. **A special case: Lagrangian interpolation.** To continue our investigation of the effectiveness of the CMPS method for approximating solutions to linear differential operator equations , we once again assume that $\Omega := K \subset \mathbb{R}^q$ is a q-dimensional cube. We also the finite-dimensional functional subspaces $\tilde{X}_N \in X$ and $\tilde{Y}_N \in Y$ for $N \in \mathbb{Z}_+$, making use of the Lagrange projection operators described above.

Let $X := L_p(K;\mathbb{R})$, $\operatorname{dom} A = W_p^{(m+s)}(K;\mathbb{R})$, $Y := L_p(K;\mathbb{R})$, $\operatorname{im} A = W_p^{(s)}(K;\mathbb{R})$, $p > q$, $s \geq 1$, and $s - q/p > 0$. It follows from the Sobolev imbedding theorem [1] that $W_p^{(m)}(K;\mathbb{R}) \subset C(K;\mathbb{R})$. So, we can construct the suitable subspaces $\tilde{Y}_N \subset Y$, $N \in \mathbb{Z}_+$, as follows:

$$
\tilde{Y}_N = P_N^{(y)} W_p^{(s)}(K;\mathbb{R}),
\tag{3.26}
$$

where the projector $P_N^{(y)} : Y \to \tilde{Y}_N$ is the classical Lagrange interpolation operator, defined as

$$
P_N^{(y)} f(x) := \sum_{(\alpha)\in\mathbb{Z}_+^q}^{N(\alpha)} f(x_{(\alpha)}) l_{(\alpha)}(x),
\tag{3.27}
$$

where $N(\alpha) := \prod\limits_{j=1}^{q} \alpha_j$, $x_{(\alpha)} \in K$ are suitable nodes in the cube $K \subset \mathbb{R}^q$, $(\alpha) \in \mathbb{Z}_+^q$ is the usual multi-index, and

$$
l_{(\alpha)}(x) := \Pi_{j=1}^q l_{\alpha_j}(x_j), \quad l_{(\alpha)}(x_{(\beta)}) = \delta_{(\alpha),(\beta)},
\tag{3.28}
$$

are basic q-dimensional Lagrange polynomials. As we have chosen the subspaces $\tilde{X}_N \in X$ and $\tilde{Y}_N \in Y$ to be finite-dimensional, condition (3.10) follows from *ii*) of Theorem 3.2. We shall need the inequality

$$
\|P_N^{(y)} \tilde{v}_N\|_{\tilde{Y}_N} \geq \tau_N^{(y)} \|\tilde{v}_N\|_Y
\tag{3.29}
$$

for all $\tilde{v}_N \in A\tilde{X}_N \in Y$, where $\tau_N^{(y)} < \infty$, $N \in \mathbb{Z}_+$. Condition (3.29) means that the projector $P_N^{(y)}$ is invertible on the subspace $A\tilde{X}_N \subset Y$, since it is evident that $\ker P_N^{(y)} = \{0\}$. Now from the definition $\tilde{v}_N = A\tilde{w}_N$, where $\tilde{w}_N \in \tilde{X}_N$, and from the fact that the operator $A : X \to Y$ is invertible on $\operatorname{dom} A$, it also follows that $\ker(P_N^{(y)} A) = \{0\}$. Additionally, we conclude from the fact that $\dim \tilde{X}_N = \dim \tilde{Y}_N$ that $\operatorname{im} (P_N^{(y)} A)|_{\tilde{X}_N} = \tilde{Y}_N$.

Consider now the subspace $C(K;\mathbb{R})$ of continuous functions on the cube $K \subset \mathbb{R}^q$ and the corresponding Lagrange interpolation projector (3.27), whose nodes are roots of Hermite's polynomials. We define

$$
\begin{aligned}
&\delta_N := \min_{1\le k\le q, 1\le j\le N_k} \{(x_k^{(j+1)} - x_k^{(j)}) : 1 \le j \le N_k\} \quad N := \Pi_{j=1}^q N_j, \\
&\lambda_N^{(y)} := ||P_N^{(y)}|| \le c_q^{(1)}(K)(\log N)^q,
\end{aligned} \tag{3.30}
$$

where $c_q^{(1)}(K) > 0$ for all $N \in \mathbb{Z}_+$. Let the functional $x^* \in C^*(K;\mathbb{R})$, $x \in K$, be defined as

$$
x^*(f) := f(x) \tag{3.31}
$$

for any function $f \in C(K;R)$. Then, as it is well known [1] for the interpolation with nodes chosen as roots of Hermite's polynomials, we have

$$
||x^* P_N^{(y)}||_{C^*(K;\mathbb{R})} = \sum_{\alpha\in\mathbb{Z}_+^q}^{N(\alpha)} |l_\alpha(x)| := \lambda_{N(\alpha)}^{(y)} \le c_q^{(2)}(K) N(\alpha)^{-s} (\log N(\alpha))^q \tag{3.32}
$$

for all $x \in K$ and $N(\alpha) \in \mathbb{Z}_+$. Whence, from Jackson's inequality [1] for each function $f \in C^{(s)}(K;\mathbb{R})$ we also have

$$
E_N[f] \le c_q^{(2)} N(\alpha)^{-s} ||f^{(s)}||_{C(K;\mathbb{R})}. \tag{3.33}
$$

where we have set $N := N(\alpha)$ to simplify the notation. Thus, making use of (3.32) and (3.33) for $f \in C^{(s)}(K;\mathbb{R})$, we obtain the estimate

$$
\begin{aligned}
&|(x^* P_N^{(y)})(f) - x^*(f)| := |P_N^{(y)} f(x) - f(x)| \le \\
&\le \sum_{\alpha\in\mathbb{Z}_+^q}^{N(\alpha)} |l_\alpha(x)|\ |f(x_\alpha) - p_N(x_\alpha) + p_N(x_\alpha) - f(x)| \\
&\le c_q^{(3)}(K)(\log N(\alpha))^q N(\alpha)^{-s} ||f^{(s)}||_{C(K;\mathbb{R})},
\end{aligned} \tag{3.34}
$$

where $c_q^{(3)}(K) := c_q^{(1)}(K) c_q^{(2)}(K) < \infty$, and p_N is the best approximating polynomial of degree $N \in \mathbb{Z}_+$ on the cube $K \subset \mathbb{R}^q$. Assuming now that differential operator (1.2) is bounded as an operator $A : C^{(s+m)}(K;\mathbb{R}) \to C^{(s)}(K;\mathbb{R})$, and for $s \ge 1$ the Kato-Rellich condition [10]

$$
||u||_{p,m+s} \le c_q^{(4)}(K)(||Au||_{p,s} + ||u||_{p,s}) \tag{3.35}
$$

holds for some constant $c_q^{(4)} > 0$ and every $u \in W_p^{(m+s)}(K;\mathbb{R})$, then employing the estimate (3.34) from the inequality (3.35), we obtain the existence of a positive constant $\tau_N^{(y)}$ such that the main inequality (3.29) holds. Accordingly it follows Theorem 3.2 that there exist a unique solution to equation (3.14) in the subspaces $\tilde{X}_N \subset W_p^{(m+s)}(K;\mathbb{R})$ for each $N \in \mathbb{Z}_+$, which approximates an exact solution to the differential equation (1.1). This means, that the system of finite-dimensional functional equations

$$
\tilde{A}_N\ \tilde{u}_N := P_q^{(y)} A P_N^{(x)} \tilde{u}_N = \tilde{f}_N := P_N^{(y)} f \tag{3.36}
$$

for all $N \in \mathbb{Z}_+$ may be represented in an equivalent form as a general system of the algebraic vector equations

$$
A_N u_N = f_N, \tag{3.37}
$$

where $u_N := \pi_N^{(x)} \tilde{u}_N$, $f_N := \pi_N^{(y)} \tilde{f}_N$, and $A_N := \pi_N^{(y)} \tilde{A}_N \pi_N^{(x),-1}$, with $\pi_N^{(x)} : \tilde{X}_N \to X_N \simeq \mathbb{R}^N$, $\pi_N^{(y)} : \tilde{Y}_N \to Y_N \simeq \mathbb{R}^N$ being the corresponding canonical isomorphisms [14, 17] between finite–dimensional subspaces for all $N \in \mathbb{Z}_+$. Thus, we have proved the following result.

Theorem 3.4. *Let the differential operator (1.2) be invertible and satisfy condition (3.35) for some $s \geq 1$. Then there exists a unique solution $u \in W_p^{(m+s)}(K;\mathbb{R})$ of equation (1.1), which is the limit of approximating solutions to finite-dimensional equations (3.37), constructed by means of the CMPS .*

As for the effective construction of finite-dimensional operators $A_N : X_N \to Y_N$, $N \in \mathbb{Z}_+$, we use the functional-algebraic properties of discrete approximations of Heisenberg-Weil algebra basis operators $\mathfrak{g}(q) := \underset{1\leq j\leq q}{\oplus}\{1, x_j, \partial/\partial x_j\}$, which were discussed above and studied extensively in [15, 14, 17]. Then expression (3.37) becomes the usual system of algebraic vector equations, whose matrix $A_N : X_N \to Y_N$, owing to the homomorphism property [23] of the universal algebra $\mathcal{U}(\mathfrak{g})$ representations, has the form

$$A_N = \sum_{|\beta|=0}^{m} a_\beta(S_N) D_N^\beta, \tag{3.38}$$

where S_N and $D_N : X_N \to X_N$ are the corresponding finite-dimensional tensor quasi-representations of basis elements (or generators) of the Heisenberg-Weil algebra $\mathfrak{g}(q)$. Solving equations (3.37) with matrices (3.38), we obtain vector solutions $u_N \in X_N \simeq \mathbb{R}^N$, which generate the desired approximate functional solutions $\tilde{u}_N := \pi_N^{(x),-1} u_N \in \tilde{X}_N \subset X$ of equation (1.1). The CMPS can also be adapted to systems with boundary and initial conditions, but we shall not pursue this matter here.

4. Illustrative Examples

We now consider a couple of examples that serve to illustrate the efficiency and effectiveness of our projective-algebraic scheme in the case of some evolution equations in partial derivatives.

4.1. **Example 1.** First, we apply the CMPS to the following linear parabolic evolution (heat) equation of the second order in two space dimensions:

$$u_t = u_{xx} + u_{yy} + t^2 \sin(x+y), \tag{4.1}$$

where $u \in C^1(0,T; W_2^2(\Omega;\mathbb{R}))$, $t \in (0,T) \subset \mathbb{R}_+$, $(x,y) \in \Omega := \{(x,y) \in \mathbb{R}^2 : \varphi(x,y) := x^2 + y^2 - 1 \leq 0\}$, with the (initial) Cauchy data

$$u_{t=0^+} = g(x,y) := 1 - x^2 - y^2, \quad (x,y) \in \Omega, \tag{4.2}$$

and the boundary condition

$$u|_{\partial\Omega} := 0. \tag{4.3}$$

The problem (4.1)-(4.3) is well posed, and we can apply our to projection-algebraic approach as delineated in the preceding sections. It is convenient to first make the following solution transformation: $u := \varphi v$, where the function $v \in C^1(0,T; W_2^2(\Omega;\mathbb{R}) \cap L_\infty(\Omega;\mathbb{R}))$ satisfies the condition (4.3). Now we take the smallest square $D := [-1,1]^2$ satisfying the condition $\Omega \subset D$. Making use of suitable mesh points with nodes $\{x^{(1)}, x^{(2)}, ..., x^{(n_x)}\} \in [-1,1]$ and $\{y^{(1)}, y^{(2)}, ..., y^{(n_y)}\} \in [-1,1]$, where $n_x, n_y \in \mathbb{Z}_+$ are sufficiently large nodal quantities.

For the projection-algebraic approximation scheme to be applied to equation (4.1), we shall use, as described above, the following Heisenberg-Weil algebra quasi-representations in finite-dimensional Euclidean vector spaces E^{n_x} and E^{n_y}, respectively:

$$x \to S^{(n_x)} := \{x^{(k)}\delta_{jk} : 1 \leq j,k = n_x\}, \quad 1 \to 1^{(n_x)} := \{\delta_{jk} : 1 \leq j,k \leq n_x\},$$

$$\partial/\partial x \to Z^{(n_x)} := \{\delta_{kj} \sum_{s\neq j}^{n_x} (x^{(j)} - x^{(s)})^{-1} + (1-\delta_{kj})(x^{(k)} - x^{(j)})^{-1} : 1 \leq j,k \leq n_x\}, \tag{4.4}$$

and

$$y \to S^{(n_y)} := \{y^{(k)}\delta_{jk} : 1 \le j,k \le n_y\}, \quad 1 \to 1^{(n_y)} := \{\delta_{jk} : 1 \le j,k \le n_y\},$$

$$\partial/\partial y \to Z^{(n_y)} := \{\delta_{kj}\sum_{s\neq j}^{n_y}(y^{(j)} - y^{(s)})^{-1} + (1-\delta_{kj})(y^{(k)} - y^{(j)})^{-1} : 1 \le j,k \le n_y\}, \tag{4.5}$$

where δ_{kj} is the Kronecker delta. These representations can be used to construct the following tensor products, acting as matrix operators in the Euclidean finite-dimensional vector space $E^{n_x \times n_y} := E^{n_x} \times E^{n_y}$:

$$\begin{aligned} &\tilde{S}_{(x)} := S^{(n_x)} \otimes 1^{(n_y)}, \quad \tilde{S}_{(y)} := 1^{(n_x)} \otimes S^{(n_y)}, \quad \tilde{1} := 1^{(n_x)} \otimes 1^{(n_y)}, \\ &\tilde{Z}_{(x)} := Z^{(n_x)} \otimes 1^{(n_y)}, \quad \tilde{Z}_{(y)} := 1^{(n_x)} \otimes Z^{(n_y)}. \end{aligned} \tag{4.6}$$

Concerning the function $v \in C^1(0,T; W_2^2(\Omega;\mathbb{R}) \cap L_\infty(\Omega;\mathbb{R}))$, one obtains easily from (4.1)-(4.3) and (4.6) the following discrete set of nonuniform linear Cauchy problems:

$$dv_n/dt = \tilde{A}_n v_n + t^2 f_n, \quad v_n|_{t=0^+} = g_n, \tag{4.7}$$

where

$$\begin{aligned} \tilde{A}_n &:= (\tilde{1} - \tilde{S}^2_{(x)} - \tilde{S}^2_{(y)})^{-1}[-4\tilde{1} - 4(\tilde{S}_{(x)}\tilde{Z}_{(x)} + \tilde{S}_{(y)}\tilde{Z}_{(y)})] + (\tilde{Z}^2_{(x)} + \tilde{Z}^2_{(y)}), \\ f_n &:= (\tilde{1} - \tilde{S}^2_{(x)} - \tilde{S}^2_{(y)})^{-1}\sin(\tilde{S}_{(x)} + \tilde{S}_{(y)})Q_n^{-1}\bar{u}_n, \quad g_n := Q_n^{-1}\bar{u}_n, \end{aligned} \tag{4.8}$$

with $\bar{u}_n := \bar{u}^{(n_x)} \otimes \bar{u}^{(n_y)}$, $\bar{u}^{(n_x)} := (1,1,...,1)^\intercal \in E^{n_x}$, $\bar{u}^{(n_y)} := (1,1,...,1)^\intercal \in E^{n_y}$, $^\intercal$ denoting the transpose, and

$$Q_n := \{\delta_{kj}\prod_{s\neq j}^{n_x}(x^{(j)} - x^{(s)}) : 1 \le k,j \le n_x\} \otimes \{\delta_{kj}\prod_{s\neq j}^{n_y}(y^{(j)} - y^{(s)}) : 1 \le k,j \le n_y\}. \tag{4.9}$$

Once we have found solutions $u_n(t) \in E^{n_x \times n_y}$, $t \in (0,T)$, to the discrete set of Cauchy problems (4.7), we obtain for all sufficiently large $n \in \mathbb{Z}_+$ the Lagrange interpolated approximate solutions

$$\tilde{u}_n(x,y;t) =< q^{(n_x)}(x) \otimes q^{(n_y)}(y), u_n(t) >_{E^{n_x \times n_y}} \in W_2^2(\Omega;\mathbb{R}), \tag{4.10}$$

where, by definition, polynomial vectors

$$q^{(n_x)}(x) := \{\prod_{s\neq j}^{n_x}(x - x^{(s)}) : 1 \le j \le n_x\}^\intercal, \quad q^{(n_y)}(y) := \{\prod_{s\neq j}^{n_y}(y - y^{(s)}) : 1 \le j \le n_y\}^\intercal, \tag{4.11}$$

for all $(x,y) \in \Omega$.

As an example, we have solved the discrete Cauchy problem (4.7) using a fourth-order Runge-Kutta scheme for $n_x = n_y = 20$, with the time interval $(0, T = 3)$ mesh taken as $h := 10^{-4}$. The resulting solutions at times $t = 0.0$, $t = 1.501$ and $t = 2.99$ are presented in Fig. 1, where the continuous curves represent isotherms ($u =$ constant level curves) for various values of time t. Here the lighter grey regions correspond to higher temperatures, and the darker grey regions indicate lower temperatures. Observe that the solution appears to be equal to zero on the straight line $\{(x,y) \in \Omega : x + y = 0\}$ - a property that can be inferred directly from the Cauchy problem (4.1)-(4.3).

4.2. **Example 2.** Next we consider the normalized heat equation in one space dimension subject to simple Cauchy data; namely:

$$u_t = u_{xx}, \quad u|_{t=0^+} := \sin x, \tag{4.12}$$

where $t \in (0,T) \subset \mathbb{R}_+$ and $x \in [-2\pi, 2\pi] \subset \mathbb{R}$. This problem has the unique bounded solution $u \in C^\infty(0,T; L_\infty(\mathbb{R};\mathbb{R}))$ of the form

$$u(x;t) = e^{-t}\sin x \tag{4.13}$$

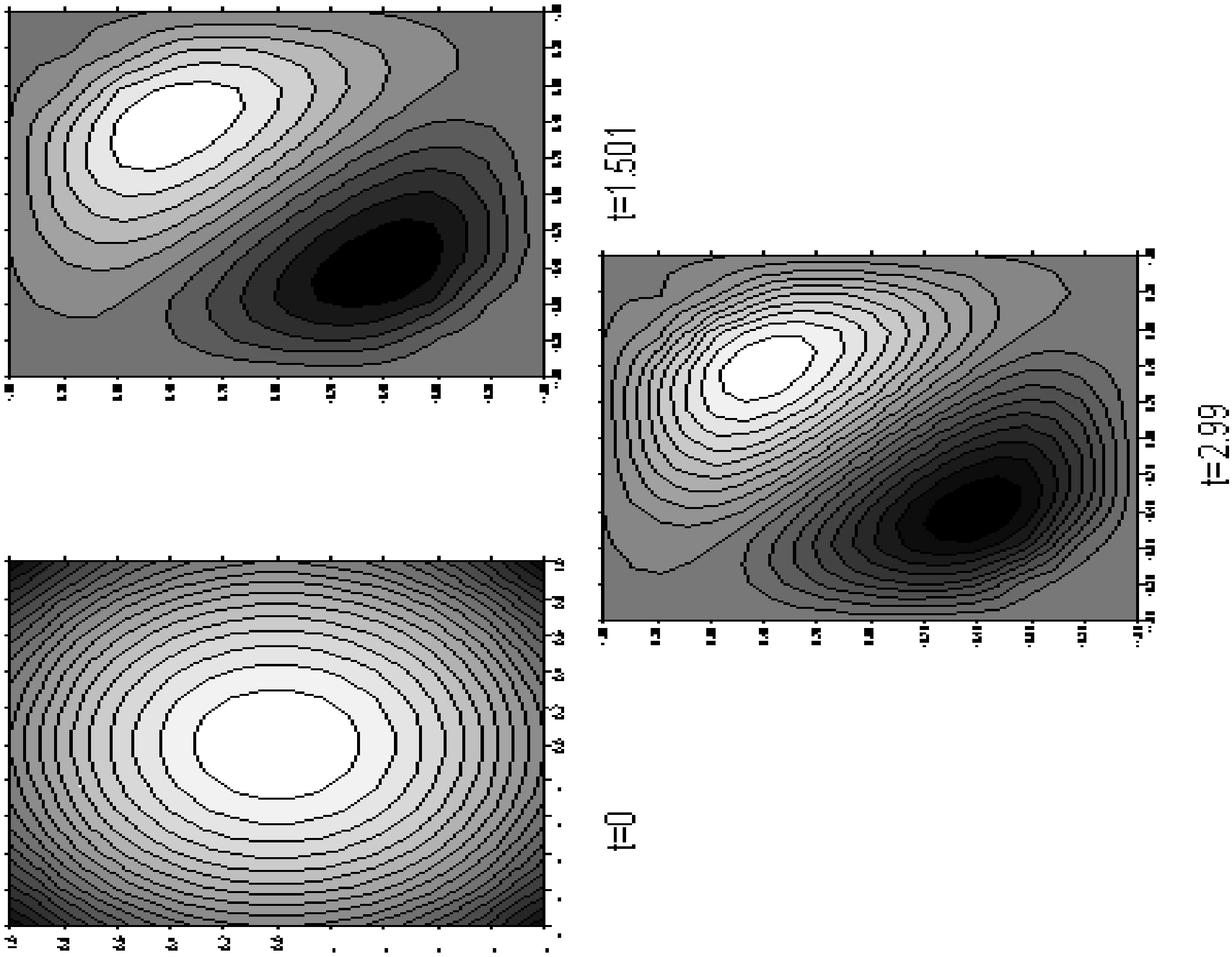

FIGURE 1. Temperature distribution in terms of isotherms at $t = 0, 1.501, 2.99$ for Example 1

for all $x \in \mathbb{R}$. The discrete approximation of (4.12) analogous to that of Example 1 in the finite-dimensional Euclidean space E^{n_x} has the form

$$du_n/dt = \tilde{Z}^2_{(x)} u_n, \quad u_n|_{t:=0^+} := g_n, \tag{4.14}$$

where $g_n := \sin(\tilde{S}_{(x)}) Q_n^{-1} \bar{u}_n = \{\sin x^{(i)} \prod_{j \neq i}^{n_x} (x^{(i)} - x^{(j)})^{-1} : 1 \leq i \leq n_x\}^{\intercal} \in E^{n_x}, \;\; n_x \in \mathbb{Z}_+$. The problem (4.14) was solved approximately using the projection-algebraic approach to obtain discrete approximations and a fourth-order Runge-Kutta method for $t \in (0, 1)$ with time interval mesh $h = 10^{-1}$ and spatial mesh obtained using $n_x = 50$. The result is presented in Fig. 2. Observe that the approximate bounded solution obtained via the CMPS, namely $\tilde{u}_n \in C^\infty(0, T; L_\infty(\mathbb{R}; \mathbb{R})), n_x \in \mathbb{Z}_+$, is very close to the exact solution (4.13) and, as expected, it is almost zero at points $\pi m \in \mathbb{Z}$, $-2 \leq m \leq 2$.

5. Concluding Remarks

In the course of this paper, we have formulated and proved precise realizability, solvability and convergence criteria for the CMPS projection-algebraic scheme for approximating solutions of (1.1) under rather mild restrictions. Moreover, we have at least provided strong indications of the effectiveness, efficiency and robustness of the CMPS scheme (implemented in algorithmic form) by applying it to an interesting nontrivial example of a system of the type (1.1). Naturally,

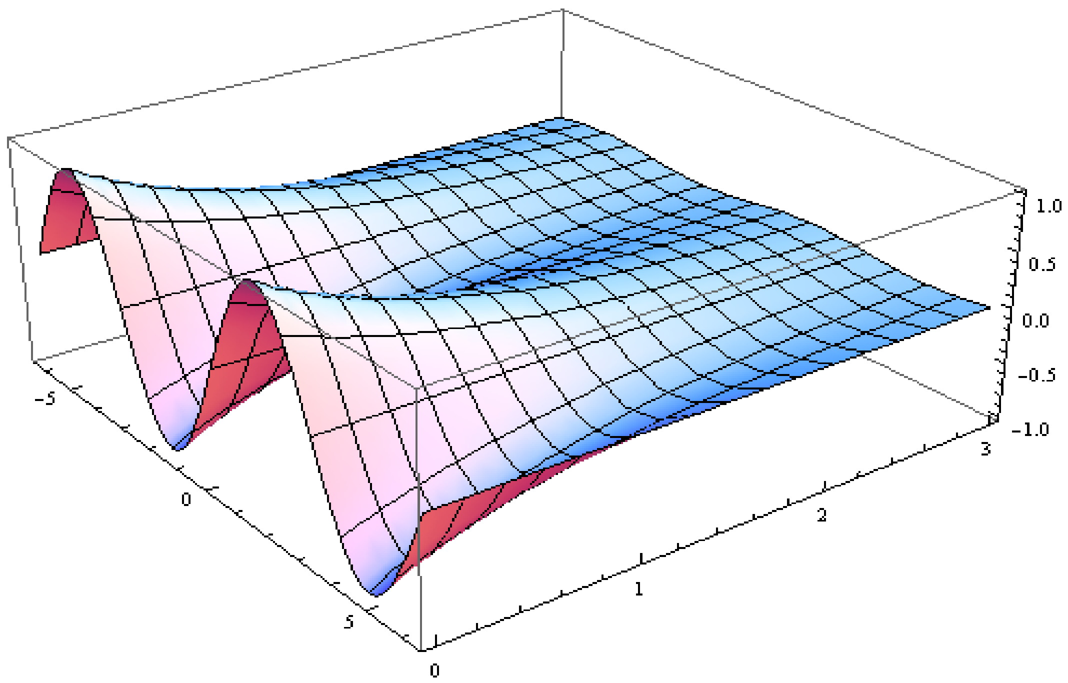

FIGURE 2. Temperature distribution for Example 2

to further demonstrate the potential this scheme as an applied tool for numerical investigation of systems arising in practice, one needs to show how it can be modified to treat approximation problems corresponding to systems (1.1) involving auxiliary specifications such as more or less standard boundary and initial conditions, and even possibly free boundary conditions. The required modifications of the CMPS approach for these types of problems, although far from obvious, are rather straightforward, and we intend to deal with them shortly in our future work. In a more dynamical vein, analogous results for the CMPS scheme can also be obtained for evolution equations in partial derivatives such as in the form $du/dt - Au = f$, where $u \in X$, $f \in Y$ and the parameter $t \in \mathbb{R}_+$. We are planning to address approximation of solutions of these types of evolution equations via the CMPS approach in a later investigation.

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